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Electron transport in mesoscopic conductors

Shepard, Kenneth Lyle, Ph.D.
Stanford University, 1992

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ELECTRON TRANSPORT IN
MESOSCOPIC CONDUCTORS

A DISSERTATION
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL ENGINEERING
AND THE COMMITTEE ON GRADUATE STUDIES
OF STANFORD UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By
Kenneth Lyle Shepard
June 1992
I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Michael L. Roukes  
(Principal Adviser)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

James S. Harris, Jr.  
(Co-adviser)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

R. Fabian Pease

Approved for the University Committee on Graduate Studies:
Abstract

This dissertation is divided into four main sections. In the first (Chapter 2), the multichannel, multiprobe Landauer formula derived by Büttiker is introduced as a general formalism for describing electron transport in mesoscopic conductors. This formula, which relates conductance coefficients to the elements of a unitary $S$-matrix, is calculated from a linear-response formalism in the presence of a uniform magnetic field. The calculation proceeds by first writing the conductance coefficients in terms of a current-current correlation function. An expansion in scattering-wave states relates this correlation function to a set of transmission coefficients that constitute a unitary $S$-matrix. The analytic properties of the $S$-matrix elements in the complex $E$ plane are used to complete the calculation with a simple contour integral. A general method for using transmission coefficients to compute four-terminal resistances is also discussed.

The second section (Chapters 3 and 4) presents a numerical technique, recursive Green's functions, for calculating the transmission coefficients through a coherent mesoscopic conductor. This is the first carefully developed extension of recursive Green's function calculations to high magnetic fields. This technique is then used to study the transition from diffusive to quantum Hall transport in quantum wires. The transition to the integral quantum Hall effect regime through the suppression of backscattering is studied theoretically on a model system by calculating the two-terminal conductance of a quasi-one-dimensional quantum wire as a magnetic field is applied in the presence of disorder. The quantum Hall regime is entered when there is negligible overlap between electron edge states localized at opposite side of the wire. Levitation of the quantized plateaus with increasing disorder is observed.
The third section (Chapter 5) considers a many-body problem. One effect which cannot be understood within the context of the independent-electron approximation is the exchange-enhanced spin splitting of Landau levels. Theoretical work is presented, suggesting that in regimes of experimental interest, the exchange enhancement of the spin splitting of Landau levels in quantum wires, as evidenced in their two-terminal conductance, is increased by the confinement. This is due to an effect in which electron-electron interactions are strengthened by interaction with polarizable edge-state charge. This antiscreening effect is confirmed by calculating exchange enhancement in a quasi-one-dimensional conductor in the “screened” Hartree-Fock approximation. Preliminary experimental evidence for the effect is also presented.

The last section of the dissertation (Chapter 6) presents the results of unique electron scattering experiments in a solid-state system. Previously, only four-terminal resistances have been measured to understand the physics of mesoscopic transport. A general experimental approach is described which yields the entire transmission matrix of a multiprobe mesoscopic conductor. Employing a sample design which is an almost literal realization of the Landauer-Büttiker model, results are presented for several new investigations with ballistic semiconductor nanostructures enabled by this technique. The first study considers the transmission coefficients for a simple cross junction, where it is found that these transmission coefficients reproduce the measured Hall and bend resistance low-field anomalies. The role of the “rebound” trajectory becomes explicit. The other investigation centers on a junction in which two of the probes are separated from the channel by quantum point contacts. This series arrangement of point contacts allows sensitive momentum spectroscopy of the emitted electron distribution. Even in the case of a single propagating mode, modal features in the outgoing distribution are observed. This indicates a non-adiabatic potential even in the presence of conductance quantization. This pinched geometry also allows the first fully characterized realization of weakly-coupled probes.
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Chapter 1

Introduction

1.1 Motivation

Conventional transistors, both bipolar and field-effect, operate on principles of classical diffusive transport, with the addition of Fermi-Dirac statistics. Transport properties are local and determined by the equations of drift and diffusion.

Technological advances have successfully pushed such conventional device designs to ever smaller dimensions with simple scaling rules. Significant technological interest in advanced epitaxial growth techniques, such as molecular beam epitaxy or silicon-on-insulator technologies, and small lithographic features, such as those defined by X-ray or electron beam, have been fueled by this trend. Nevertheless, as devices are eventually scaled to 0.1\(\mu\text{m}\) and below, the laws of classical diffusive transport must inevitably break down. Efforts should then be directed either toward using the resulting ballistic or quantum-size effects to produce novel device operation\cite{1} or, at the least, toward understanding them as “parasitics” on conventional transistor action. With this engineering motivation in mind, this thesis makes contributions to the understanding of the fundamental transport properties of semiconductor nanostructures. This work fits into the general field of research in mesoscopic systems.\cite{1}

\footnote{For a more complete review of mesoscopic physics, see Beenakker and van Houten\cite{2}.}
1.2 Mesoscopic systems

This dissertation considers transport in the limit of linear response. The voltages applied are small enough that current is linear with voltage with a constant of proportionality given by the conductance, \( g \). For a macroscopic homogeneous conducting system\(^2\), a conductivity tensor \( \sigma \) relates the current density and the electric field, \( \vec{j}(\vec{r}) = \sigma \vec{E}(\vec{r}) \). If the medium is also isotropic, then the conductivity tensor is diagonal and \( \vec{j}(\vec{r}) = \sigma \vec{E}(\vec{r}) \). These equations contain the assumption of locality. The current density at a point \( \vec{r} \) depends only on the electric field at that point \( \vec{r} \). This requires that the electric field not vary appreciably over distances comparable to the electron mean free path; that is, the distance between collisions completely effective in returning electrons to a thermal distribution. The conductance\(^3\), which determines the relationship between the total current and the voltage drop across the sample, is related to the conductivity in macroscopic systems by a simple geometric scaling in the homogeneous, isotropic case (see Figure 1):

\[
g = \begin{cases} 
\frac{\sigma A}{L} & \text{three dimensions} \\
\frac{\sigma W}{L} & \text{two dimensions} \\
\frac{\sigma}{L} & \text{one dimension}
\end{cases} \tag{1}
\]

where \( A \) is the cross sectional area of the sample perpendicular to the direction of (uniform) current flow for three dimensions, \( W \) is the width of the sample perpendicular to the direction of current flow for two dimensions, and \( L \) is the length of the sample in the direction of current flow. Note that in two dimensions, \( \sigma \) and \( g \) have the same units.

When the size of the sample, the length scale for variation in the electric field in the linear-response regime, becomes comparable with the elastic or inelastic scattering length, the assumption of locality breaks down. In this mesoscopic regime, the

---

\(^2\)This thesis will be concerned only with metallic systems, that is systems in which the localization length of the electron wavefunction \( \xi \) (at zero magnetic field) is large compared to the sample size, \( \xi \gg L \). Problems associated with the insulator and the nature of the metal-insulator transition are contained in review papers on disordered electronic systems\(^3\).

\(^3\)In the linear-response regime, the conductance is independent of the details of the spatial variations in electric field and current density\(^4\). This will become more clear in Section 2.2.
Figure 1: Scaling of electron transport. For macroscopic systems, the conductance $g$ scales geometrically with the local conductivity $\sigma$. This scaling breaks down in the mesoscopic regime when sample dimensions become comparable with some characteristic length scale, such as the elastic mean free path or phase coherence length. The microscopic regime is on the level of individual unit cells of the crystal lattice.
macroscopic scaling behavior of Equation 1 no longer applies. The term *mesoscopic* places dimensions between the *macroscopic* regime and the *microscopic* regime, which occurs at the level of the individual unit cells of the crystal lattice (see Figure 1). In mesoscopic conductors, the conductance rather than the conductivity is fundamental, and transport properties depend on the details of the interaction of the electrons with the current and voltage probes. *Nonlocality* implies that scatterers outside the net current path contribute to conductances. The Landauer-Büttiker formula introduced in Chapter 2, which treats electrical transport as a scattering problem, has become central to understanding electrical conductance in nonlocal systems. Research in mesoscopic structures can be roughly categorized into one of three principal regimes: quantum diffusive transport, ballistic transport, and high-field transport.

1.2.1 Quantum diffusive transport

If the sample size is much larger than the elastic mean free path \((W, L \gg l)\), then transport is diffusive and characterized by a diffusion constant, \(D\).\(^4\) Elastic scattering randomizes momentum but conserves energy. Scattering in this case can be due to impurities, interface roughness, or alloy disorder. Si MOSFETs\(^5\) and thin metal films are two examples of diffusive conducting systems. In contrast to elastic processes, inelastic processes, which include electron-phonon and electron-electron interactions\(^6\), cause electrons to decay from energy eigenstates and lose phase memory.

When a diffusive submicron sample is cooled to low temperatures, the phase coherence length, or inelastic scattering length, \(L_{in} = (D\tau_{in})^{1/2}\) can become quite large in relation to the sample dimensions, where \(\tau_{in}\) is the inelastic relaxation time. Several interesting phenomena occur in this phase-coherent (quantum) diffusive regime (see Figure 2a):

- *Weak localization* is observed even in samples that are quite large with respect to \(L_{in}\) and is usually treated as a correction to the local Drude conductivity due

\(^4\)In two dimensions, \(D = \frac{1}{2} v_F l\) where \(l\) is the elastic mean free path and \(v_F\) is the Fermi velocity.

\(^5\)Both of these scattering processes are suppressed at low temperature. Fermi-Dirac statistics limit the available phase space for scattering. In addition, at low temperature, such scattering events are small and are relatively ineffective in randomizing momenta.[6]
Figure 2: Three regimes of investigation in mesoscopic systems. (a) *Quantum diffusive transport*. Sample dimensions \((W, L)\) are much larger than the elastic mean free path \(l\) but smaller than the inelastic scattering length \(L_{in}\). (b) *Ballistic transport*. Sample dimensions are smaller than the elastic mean free path so that electron dynamics are determined by the engineered boundaries of the sample. (c) *High-field transport*. Electron transport is confined to edge states when the Fermi energy lies far enough away from bulk Landau-level energies. These edge states correspond to classical skipping orbits.
to coherent backscatter[7]. The introduction of a magnetic field destroys time-reversal invariance and this coherent superposition. The Drude conductivity is restored when the magnetic length \( l_B = \left( \frac{\hbar}{eB} \right)^{1/2} \) becomes comparable to \( L_{in} \). Weak localization is, therefore, characterized by a peak in the magnetoresistance around zero field, where the extent of the peak gives an estimate of \( L_{in} \). A good review of this phenomena is provided by Bergmann[8].

- Phase-coherent conductors in the regime of diffusive transport smaller than \( L_{in} \) show fluctuations in the conductance as a function of magnetic field or Fermi energy of magnitude \( e^2/h \) (at zero temperature), independent of the size of the sample and amount of disorder[9]. These universal conductance fluctuations (UCF) are completely reproducible and are sometimes referred to as the "fingerprint" of the sample. When the sample size exceeds the phase-coherence length (\( L > L_{in} \)), the fluctuations, \( \Delta g/g \), are averaged out over \( L/L_{in} \) phase-coherent segments of the sample[10].

- While UCF are aperiodic, periodic fluctuations are observed in ring geometries smaller than \( L_{in} \) because of a well-defined enclosed magnetic flux \( \Phi \) within the ring. Aharonov-Bohm oscillations of period \( \frac{\hbar}{e} \) and \( \frac{\hbar}{2e} \) in the magnetic field are observed[11]. An excellent review is given by Washburn and Webb[12].

### 1.2.2 Ballistic transport

While diffusive transport is characterized by dimensions much larger than the elastic mean free path, *ballistic transport* is achieved when the sample size is less than the elastic scattering length. In this regime, the conducting system must be extraordinarily clean and scattering is due only to the engineered potentials of the sample (see Figure 2b).

If the temperature is high enough (\( \sim 2K \)) that there is enough small-angle inelastic scattering (due to electron-electron interactions and acoustic phonon scattering) to destroy phase coherence while having little randomizing effect on momentum, then the electrons act much like classical billiards. The work in this regime has centered on magnetotransport within junctions[13, 14, 15], focussing geometries[16], or periodic
lattices[17]. Chapter 6 presents the results of unique scattering experiments in ballistic devices.

If the temperature is low enough, the electron transport will not only be ballistic but phase-coherent. Complicated conductance fluctuations associated with constructive and destructive interference of electron waves will now be superimposed upon the classical behavior.

It is also possible to study samples which are so small that they confine electrons on the scale of the Fermi wavelength. In this case, quantum size effects become important. One of the most dramatic phenomena associated with few-mode transport is the quantized resistance of very short quasi-one-dimensional ballistic wires[18, 19]. The two-terminal conductance of these quantum point contacts is found to be quantized in units of $2e^2/h$, with each propagating quasi-one-dimensional mode contributing one such unit to the conductance. The quantum point contact potential is studied in considerable detail in Chapter 6.

### 1.2.3 High-field transport

With the discovery of the quantum Hall effect[20, 21], there has been considerable interest in two-dimensional electron gases (2DEG’s) in the presence of strong perpendicular magnetic fields. The energy spectrum is quantized into a series of Landau levels. When the Fermi level lies far enough away from these bulk Landau-level energies, one obtains few-mode quantum transport over macroscopic length scales. For macroscopic samples of dimensions larger than $L_{in}$, transport is local and defined by a two-dimensional resistivity tensor.

$$
\rho = \begin{pmatrix}
\rho_{xx} & \rho_{xy} \\
\rho_{yx} & \rho_{yy}
\end{pmatrix}
$$

(2)

The integral quantum Hall effect is characterized by a Hall resistivity quantized as

$$
\rho_{xy} = \frac{\hbar}{e^2\nu}
$$

(3)

where $\nu$ is the number of occupied (spin-split) Landau levels, and by a vanishing longitudinal resistivity,

$$
\rho_{xx} = \rho_{yy} = 0
$$

(4)
When the Fermi level approaches the bulk Landau-level energy, $\rho_{xx}$ becomes finite and $\rho_{xy}$ assumes intermediate values between the quantized levels.\footnote{Figure 30 shows $\rho_{xx}$ and $\rho_{xy}$ for the 2DEG studied in Chapter 6.} In samples in which one or more dimensions become comparable with the inelastic scattering length $L_{in}$, one can no longer define such a local resistivity tensor. Resistances and conductances become fundamental in this mesoscopic regime and are calculated using the Landauer-Büttiker formula as discussed in Section 2.4. Transport is described in terms of Fermi-level states confined to the edges of the sample. These correspond to the classical skipping orbits shown in Figure 2c. In this picture, the quantum Hall effect occurs because of the suppression of edge-state backscatter in the presence of a magnetic field\cite{22}. Edge states in quasi-one-dimensional systems and the transition between the quantum diffusive and high-field regimes are studied in detail in Chapter 4.

In a sufficiently disordered system, there is a significant amount of inter-Landau-level scattering (i.e. scattering between edge states on the same side of the sample). In experiments on very high mobility 2DEG's, however, potentials can be smooth on the scale of the magnetic length $l_B (\sim 10nm$ at $B = 5T$). Transport in this case is said to be adiabatic; potentials are slowly-varying enough that mode number is preserved across the dimensions of the device and inter-Landau-level scattering is strongly suppressed\cite{23, 24}. Mesoscopic phenomena can be observed at length scales approaching $\sim 80\mu m$ or more. These include the breakdown of macroscopic scaling of the longitudinal conductance\cite{25} and the anomalous integer quantum Hall effect\cite{23, 26}, in which point contact probes are used to selectively populate and detect edge channels in the integral quantum Hall effect regime. Adiabatic transport in quantum wires will be discussed in Chapter 4.

The magnetic field also couples to the spin degrees of freedom of the electrons. This results in spin splitting of otherwise degenerate energy levels. This spin splitting can be enhanced by many-body interactions, an effect studied in Chapter 5 in the presence of quasi-one-dimensional confinement.
1.3 Ballistic and adiabatic systems in semiconductor nanostructures

1.3.1 The AlGaAs/GaAs 2DEG

The material system of choice for ballistic and high-field transport studies is the two-dimensional electron gas (2DEG) that exists at the heterointerface in a modulation-doped $\text{Al}_x\text{Ga}_{1-x}\text{As}/\text{GaAs}$ heterostructure[27]. This semiconductor system is generally grown by molecular beam epitaxy (MBE) [28] and has the basic structure shown in Figure 3a. Typically, an undoped $\text{GaAs}$ buffer of $\sim 1\mu m$ or more is grown on a $\text{Cr}$-doped semi-insulating $\text{GaAs}$ substrate. This is followed by an undoped $\text{Al}_x\text{Ga}_{1-x}\text{As}$ spacer layer and a $\text{Si}$-doped $\text{Al}_x\text{Ga}_{1-x}\text{As}$ layer. For aluminum contents higher than
Figure 4: The two-dimensional electron gas (2DEG) is the starting material for microfabrication-imposed potentials in mesoscopic structures.

$x = 0.3$, it becomes difficult to dope the AlGaAs. Growth is terminated with a GaAs cap layer, typically 50Å and doped with Si. The cap layer facilitates the formation of good Ohmic contacts to the 2DEG. Sometimes a GaAs/AlGaAs superlattice buffer is grown before the undoped GaAs buffer. This has been shown to help trap impurities outdiffusing from the substrate[29].

The corresponding band diagram for the modulation-doped heterostructure is shown in Figure 3b. The Si donors in the AlGaAs are ionized and the electrons spill into the triangular potential well created by the conduction band offset ($\Delta E_c \sim 0.3\text{eV}$) between the wider bandgap $Al_{0.3}Ga_{0.7}As$ and the narrower bandgap GaAs. The confinement perpendicular to the interface produces two-dimensional subbands within the triangular potential well. The first subband is approximately 20-40 meV above the conduction band and the separation between the first and second subbands is about 30 meV. For typical sheet densities of $1 - 5 \times 10^{11}\text{cm}^{-2}$, only the lowest subband is occupied at low temperatures (below 40 K) and free motion is completely
confined to the plane of the interface. At room temperature, however, several subbands are occupied, even at low carrier concentrations.

The undoped spacer acts to separate the 2DEG from the ionized donors. This, along with the extraordinarily smooth heterointerface, produces an exceptionally clean conducting system. Impurity-limited mobilities as high as $1.1 \times 10^7 \text{cm}^{-2}/\text{V} - \text{sec}$ have been reported[30]. In addition, the simple circular Fermi surface ($m^* = 0.067m_e$) and the reduced dimensionality help to make this an ideal system for fundamental transport studies. Because the density is low in these systems, the Fermi wavelength can also be very long ($\sim 400 - 1000 \text{Å}$), so that quantum size effects are easily observed. A comprehensive review of the physics of the 2DEG has been given by Ando, Fowler, and Stern[31]. The materials properties of the $GaAs/AlGaAs$ material system have been reviewed by Adachi[32].

### 1.3.2 Lateral confinement of the two-dimensional electron gas

With this high-mobility 2DEG as the starting material, lateral confinement techniques are introduced to carve the conductor up into various geometries for study as shown in Figure 4. Lateral confinement is achieved by selective depletion of the 2DEG, and several different schemes have been developed to accomplish this.\(^7\) The simplest of these is a simple wet or dry etch of the semiconductor material, producing “mesas” of 2DEG. It turns out that only partial removal of the doped $AlGaAs$ is sufficient to deplete the electron gas. Etching is a very invasive procedure that introduces sidewall damage in the remaining 2DEG and is, therefore, primarily used to define only large isolated regions for further patterning.

Two other techniques have been developed which, along with electron beam lithography, are used to define the smallest features under study. The first of these, developed by M. L. Roukes and A. Scherer at Bellcore[34], uses a low-energy ion exposure to enhance the surface potential and deplete the underlying 2DEG. The technique uses noble gas ions (principally Ne) at energies from 100 - 500 eV. A second high

\(^7\)For a more complete review of confinement schemes, see Reed[33].
resolution confinement scheme, the so-called split-gate lateral confinement technique, was developed by Thornton, et al.[35] and Zhang, et al.[36]. This technique uses a Schottky gate which is biased negatively with respect to the 2DEG to deplete the electron gas from the region beneath the gate. The nature of this charge control has been well-studied as it applies to modulation-doped field-effect transistors[37].

All three of these techniques are used to produce the mesoscopic structures studied in Chapters 5 and 6.

1.4 Outline

In Chapter 2, the Landauer-Büttiker formula, a formalism in which conductance is related to transmission and reflection properties, is discussed in some detail. This scattering formalism is central to current understanding of nonlocal transport in mesoscopic systems. This chapter presents a unique linear-response calculation of the Landauer-Büttiker formula, which utilizes the analytic properties of the microscopic $S$-matrix. In Chapter 3, a numerical technique, recursive Green’s functions, is presented for calculating the transmission coefficients through a coherent mesoscopic conductor. This represents the first complete exposition of the technique to high magnetic fields. In Chapter 4, recursive Green’s functions are utilized to study the transition from diffusive to quantum Hall transport in quantum wires.

Most of the mesoscopic phenomena studied can be understood within the context of the independent-electron approximation. One effect which cannot is the exchange-enhanced spin-splitting of Landau levels. In Chapter 5, this many-electron effect is examined in the presence of quasi-one-dimensional confinement. It turns out that electron-electron interactions, which give rise to the exchange enhancement, are strengthened (that is, “antiscreened”) rather than screened by the presence of polarized edge-state charge.

Finally, in Chapter 6, a definitive study of classical ballistic transport in junctions is presented by introducing a technique to measure the transmission coefficient of ballistic nanostructures experimentally. The technique is also applied to a pinched geometry which allows a very sensitive momentum spectroscopy of the electrons emitted
from a few-mode quantum point contact. The recursive Green's function technique of Chapter 3 is used to understand the details of the quantum point contact potentials producing the modal features observed in the emitted distribution.
Chapter 2

Landauer-Büttiker Formula

2.1 Introduction

Nonlocality is one of the hallmarks of mesoscopic transport. Resistances and conductances, rather than resistivities and conductivities, are fundamental. The Landauer-Büttiker formula has become an important tool for understanding electron transport in this case.

Over twenty years ago, Landauer[38, 39] showed that the DC conductance of a one-dimensional system of non-interacting electrons\(^1\) in a disordered medium is given by \( g = \frac{e^2}{h} \frac{T}{R} \) per spin degree of freedom, where \( T \) and \( R \) are the transmission and reflection probabilities, respectively. The derivation of Landauer is easily extended to quasi-one-dimensional systems (i.e., those with several occupied subbands, or modes) with multiple probes[41]. These formulae have the property that a dissipationless channel (i.e., a mode that is transmitted through the conductor without scattering) has zero resistance. In this case, it is argued that the conductance is given by dividing the current by the "local chemical potential" difference, which is smaller than the real chemical potential difference because of a nonequilibrium distribution of carriers in the leads resulting from self-consistent screening. Generalized multichannel versions of this formula have also been derived from the Kubo formula of linear response[42, 43].

More recently, the conductance has been calculated by dividing the current by

\(^1\)These non-interacting electrons are actually the quasiparticles of the Fermi liquid state[40].
the actual difference in the chemical potentials, determined by the particle densities well within the phase randomizing reservoirs\cite{44}. For the two-probe case, this gives $g = \frac{e^2}{h}T$ per spin degree of freedom, which predicts a finite contact resistance for a dissipationless channel. There has been considerable controversy over these two competing forms of the Landauer formula.\footnote{For a perspective on the controversy, see Stone and Szafer\cite{45}.} Nevertheless, this second theoretical abstraction, in which measurements are performed from reservoir to reservoir and the Landauer formula takes the $g = \frac{e^2}{h}T$ form, has been remarkably successful in describing experimental results, and the controversy has largely been resolved.

With this second definition of conductance, Büttiker has provided a general framework for treating transport in coherent mesoscopic structures as a scattering problem. Such a generalized scattering structure as microfabricated in a 2DEG is shown in Figure 5. Disorder is confined to the scattering region, which is fed by quasi-one-dimensional leads, each of which has a finite number of occupied transverse states, or modes. An incoming electron in mode $m$ of lead $i$ is scattered into mode $n$ of lead $j$ with transmission amplitude $t_{ij,mn}$. The $t_{ij,mn}$ form part of a unitary multiprobe, multichannel $S$-matrix with elements determined by elastic scattering from the scattering potential, \textit{i.e.} by the microfabricated sample geometry and disorder. The Landauer-Büttiker formula provides a framework for relating current and voltages to these amplitudes within the context of linear response\cite{44}. Current is fed into each lead by a reservoir at chemical potential $\mu_i$. The reservoirs are assumed to contact the leads without introducing any additional sources of scattering\cite{46}. All voltage drops occur as the thermal equilibrium distribution of electrons in the reservoirs is injected into or collected from the quasi-one-dimensional leads\cite{47}. The multichannel, multiprobe version of the Landauer-Büttiker formula is given by:

$$\frac{h}{2e}I_i = -N_i\mu_i + \sum_j T_{ij}\mu_j$$

(5)

where $i$ and $j$ are indices labelling the leads. The $T_{ij}$ are the transmission coefficients, given by the transmission probabilities evaluated at the Fermi energy and summed over channel indices:

$$T_{ij} = \sum_{mn} |t_{ij,mn}(E_F)|^2$$

(6)
Figure 5: Generalized scattering structure microfabricated in a 2DEG. As an example, the transmission amplitude $t_{ij,mn}$ corresponds to an incoming electron in mode $m$ of lead $i$ is scattered into mode $n$ of lead $j$. Each lead has a finite number of occupied transverse states, or modes. Current is fed into each lead by a reservoir at chemical potential $\mu_i$. 


These represent the total transmission (i.e. number of effective propagating modes) from all occupied modes in lead $j$ to all occupied modes in lead $i$. $N_i$ is the number of propagating modes in the lead $i$; $N_i = \sum_j T_{ij}$. Positive currents imply flow into the leads. The factor of two accounts for the spin degeneracy of each channel. If the quantum modes are spin-split, as in the presence of a magnetic field as discussed in Chapter 5, then the prefactor $\hbar/2e \rightarrow \hbar/e$

The voltages measured at each reservoir are related to the chemical potentials by the electronic charge $V_j = \mu_j/e$. At finite temperature, the transmission coefficients are weighted by the derivative of the Fermi function:

$$T_{ij} = \sum_{mn} \int dE \left(-\frac{\partial f}{\partial E}\right) |t_{ij,mn}(E)|^2 \tag{7}$$

Time-reversal invariance implies that the microscopic transmission amplitudes satisfy

$$t_{ij,mn}(B) = t_{ji,mn}^*(-B) \tag{8}$$

This implies that the transmission coefficients satisfy the reciprocity relation[44].

$$T_{ij}(B) = T_{ji}(-B) \tag{9}$$

Equations 5 and 9 are derived under the assumption of quantum-mechanical phase coherence. Büttiker has shown, however, that phase-breaking (inelastic) scatterers can be modelled by the introduction of additional reservoirs, through which one constrains the current to be zero. Because the addition of reservoirs has no effect on the transmission coefficients or their symmetry properties[48], the Landauer-Büttiker formula and the reciprocity relations apply equally well to situations in which the electron dynamics are incoherent but otherwise deterministic, as in the case of classical ballistic transport (discussed in Chapter 6) or transport in the presence of high magnetic fields from an edge-state picture[49] (see Section 2.4).

### 2.2 Linear response derivation

It is important to connect equation 5 with traditional linear-response calculations of conductance based on the Kubo formula[50, 45, 51, 52]. The elegance of the Kubo
formula is that it relates transport coefficients in the limit of linear response to the
equilibrium properties of the system.\textsuperscript{3}

This section presents a simple, alternate derivation of the conductance coefficients
in the presence of a uniform magnetic field from linear-response theory[59]. The for-
mulation includes the explicit cancellation of the electron velocity in the product of
the one-dimensional density of states and the current and, therefore, carries some of
the intuition of Büttiker's original work[44]. The calculation proceeds by first writing
the conductance coefficients in terms of a current-current correlation function. An
expansion in scattering wave states relates this correlation function to a set of trans-
mission coefficients that constitute a unitary $S$-matrix. The calculation is completed
with a simple contour integral.

Consider a generalized multiprobe geometry similar to that considered by Stone
and Szafer[45] as shown in Figure 6 for the specific case of three leads. The probes
extend as ideal leads to infinity, implying a continuous energy spectrum and irre-
versibility. The electric field is zero in the asymptotic regions beyond some lines $P_i$
in each lead $i$. The potential $V_i$ in the asymptotic regions of each of the leads is
constant in space, varies in time with frequency $\omega$, and has a value which corresponds
to the chemical potential of the reservoir associated with that lead, $\mu_i = e V_i$. The
zero-frequency limit is taken at the end of the calculation. This system is assumed
to model that of a finite system with phase-randomizing reservoirs in the leads by
arguments well represented in the literature[50, 45, 51]. The reservoirs in this model
are perfectly coupled to the ideal leads, adding no sources of additional scattering[46].
A uniform time-independent magnetic field exists perpendicular to the plane of the
multiprobe geometry. Disorder is assumed confined to the region of finite electric

\textsuperscript{3}Recent experiments demonstrate that the regime of linear response may be quite limited, par-
ticular in the case of phase-coherent conductors[53, 54]. However, the more general problem of
transport in the presence of arbitrarily strong accelerating fields is considerably more difficult. In
this case, the problem is best treated by a non-equilibrium Green's function theory, as developed by
Kadanoff and Baym[55] and Keldysh[56], a formalism which produces rather intractable equations
of motion for the propagators. A heuristic approach extending the Landauer-Büttiker formula to
nonlinear transport by using the energy dependence of the transmission coefficients provides only
qualitative connection to experiment[57, 58].
Figure 6: Generalized multiprobe geometry shown for the case of three leads. The electric field is assumed to be zero well within the leads, beyond some planes $P_i$ in each lead $i$. A local coordinate system $(x_i, y_i)$ is defined in each lead.
field. A local coordinate system \((x_i, y_i)\) is defined in each lead. Furthermore, the confining potential of the ideal leads is assumed to have a minimum centered at \(y_i = 0\) and to be a monotonically increasing function of \(|y_i|\).

2.2.1 Scattering wave state basis for the multiprobe geometry

Define an electron field operator \(\hat{\Psi}(x, y)\) for electrons in the multiprobe geometry. For simplicity, the electrons are assumed spinless. The electromagnetic current density in the asymptotic region of lead \(i\) in the presence of a magnetic field is given by:

\[
J_i(x_i, y_i) = -\frac{e\hbar}{2mi} \left[ \hat{\Psi}^\dagger \nabla \hat{\Psi} - (\nabla \hat{\Psi}^\dagger) \hat{\Psi} \right] - \frac{e^2}{mc} A_o \hat{\Psi}^\dagger \hat{\Psi}
\]

(10)

where \(\nabla \times A_o = B_z \hat{z}\), the uniform magnetic field in the \(z\)-direction. In the region of finite electric field, the total magnetic vector potential is given by \(A(x, y, t) = A_o(x, y) + \delta A(x, y, t)\), where \(\delta A(x, y, t)\) is associated with the applied electric field induced by the potentials in the asymptotic regions of the leads, \(E = -\frac{1}{c} \frac{\partial A}{\partial t}\), with the scalar potential chosen to be zero in the finite field region.

Choosing the Landau gauge \(A(x, y) = (-yB, 0)\), the eigenstates of the ideal lead for noninteracting fermions are given by,

\[
\psi_{nk}(x, y) = \frac{1}{\sqrt{L}} e^{\pm ikx} \chi_{nk}(y)
\]

(11)

where \(n\) is the index of the transverse state and \(L\) is the length scale for box normalization. The plus and minus signs denote outgoing and incoming states, respectively, and \(k\) is explicitly positive. \(\chi_{nk}(y)\) is chosen so that \(\int dy |\chi_{nk}(y)|^2 = 1\). The transverse state is independent of \(k\) in the absence of a magnetic field. In the presence of a magnetic field perpendicular to the plane of the wire, the Lorentz force acts to spatially separate electrons moving in opposite directions in \(x\). The properties of these lead eigenstates and the associated \(E - k\) relations are discussed further in Section 4.3. The field operators can be expanded in scattering wave states, which at a given energy, can be considered complete in the multiprobe Hilbert space.\(^4\) At a

\(^4\)The strict sense in which these states can be considered complete and the complications introduced by bound or quasi-bound states are discussed in standard books on scattering theory such as Goldberner and Watson[60].
given energy, $E$, the stationary scattering wave states corresponding to an incoming electron in lead $j$ in channel $\alpha$ is given in the asymptotic regions of leads $j$ and $i \neq j$ by:

$$
\phi_{j,\alpha}^+(x, y) = \begin{cases} 
\psi_{\alpha k}^-(x_j, y_j) + \sum_\nu t_{ij,\nu,\alpha}(E) \sqrt{\frac{\hbar \omega}{2m}} \psi_{\nu k}^+(x_j, y_j) & \text{in lead } j \\
\sum_\nu t_{ij,\nu,\alpha}(E) \sqrt{\frac{\hbar \omega}{2m}} \psi_{\nu k}^+(x_i, y_i) & \text{in lead } i
\end{cases}
$$

(12)

The velocities are given by:

$$
|v_\alpha| \equiv \frac{1}{\hbar} \frac{dE'}{dk} = \frac{\hbar}{m} \left[ k - \frac{eB}{\hbar c} \int \chi_{\alpha k}^+ y \chi_{\alpha k}^+ dy \right] 
$$

(13)

The $t_{ij,\nu,\alpha}(E)$ as introduced in Equation 12 are elements of a unitary multichannel, multiprobe $S$-matrix. In this notation, the first two indices refer to the leads and the second two indices refer to the transverse channels in these leads. The time-ordering of the incoming and outgoing lead states gives the $t_{ij,\nu,\alpha}(E)$ certain properties when analytically continued into the complex-$E$ plane. Specifically, $t_{ij,\nu,\alpha}(E)$ is analytic everywhere in the upper half-plane, and $t_{ij,\nu,\alpha}^*(E)$ is analytic everywhere in the lower half-plane. The $\sqrt{|v_\alpha|/|v_\nu|}$ normalization factors are required to account for the different amounts of flux per unit probability density carried by each channel, rendering flux conservation tantamount to $S$-matrix unitarity[50].

### 2.2.2 Conductance and the current-current correlation function

The DC current through the $i$th lead is given by:

$$
I_i = \int_{P_i} dy_i J_i(x_i, y_i) 
$$

(14)

$$
= \int_{P_i} dy_i \int_{\mathcal{R}} dx' \, dy' \, \hat{e}_i \cdot \sigma(x_i, y_i, x', y') \mathbf{E}(x', y')
$$

(15)

$\sigma$ is the conductivity tensor. $J_i$ is now explicitly in the $\hat{e}_i$ direction for each lead. $\mathcal{R}$ is the entire region of non-zero electric field of the multiprobe structure. Integrating by parts after expressing $\mathbf{E}$ as a gradient of a scalar potential $V$ and using the results of current conservation[45, 51],

$$
I_i = \sum_j g_{ij} V_j = - \sum_j \int_{P_i} dy_i \int_{P_j} dy_j \sigma_{ij}(x_i, y_i, x_j, y_j)V_j
$$

(16)
where \( g_{ij} \) are the conductance coefficients and the \( V_j \) are the voltages in the ideal leads. The conductivity evaluated between points in the asymptotic regions of two different leads is related to the current-current response function for these two leads[61],

\[
\begin{align*}
\sigma_{ij}(x_i, y_i, x_j, y_j, \omega) &= \int_{-\infty}^{\infty} dt \, e^{i\omega t} \, R_{ij}(x_i, y_i, x_j, y_j, t) \\
R_{ij}(x_i, y_i, x_j, y_j, t) &= -i\Theta(t)[J_i(x_i, y_i, t), J_j(x_j, y_j, 0)] \\
&\quad - ne^2 \frac{m}{\hbar^2} \delta(x_i - x_j) \delta(y_i - y_j) \delta(t)
\end{align*}
\]  

(17)

(18)

\( \sigma_{ij} \) and \( R_{ij} \) represent the appropriate tensor components corresponding to the \( \hat{x}_i \) and \( \hat{x}_j \) directions. \( n \) is the areal electron density. The angle brackets denote a thermal average \( \ldots = Tr \, e^{-\beta(H - \mu N)} \ldots / Z \) where \( Z = Tr \, e^{-\beta(H - \mu N)} \), \( N \) is the number operator, and \( \mu \) is the chemical potential.

Explicitly taking \( i \neq j \) for the remainder of the discussion, the off-diagonal conductance coefficients are given by:

\[
\sigma_{ij} = \lim_{\omega \to 0} \frac{1}{\omega} \int_0^\infty dt \int_{P_i} dy_i \int_{P_j} dy_j \, e^{i\omega t}[J_i(x_i, y_i, t), J_j(x_j, y_j, 0)]
\]

(19)

One observes that the diamagnetic piece of the response gives no contribution to the conductance between two different leads.

### 2.2.3 Expanding the correlation function in the scattering wave basis

Using Equation 10, the current-current correlation function can be expanded in the scattering wave states introduced in Equation 12 at energies \( E \) and \( E' \). The state at energy \( E' \) corresponds to an incoming electron in lead \( m \), transverse state \( \alpha \), with longitudinal momentum \( k \):

\[
\phi_{m\alpha}^+(x_i, y_i) = \begin{cases} 
\sum_\nu t_{im, \nu\alpha}(E') \sqrt{\frac{\hbar \omega_n}{\mu \lambda}} \psi_{\nu k'}^+(x_i, y_i) & \text{if } i \neq m \\
\psi_{\alpha k}(x_i, y_i) + \sum_\nu t_{ii, \nu\alpha}(E') \sqrt{\frac{\hbar \omega_n}{\mu \lambda}} \psi_{\nu k'}^+(x_i, y_i) & \text{if } i = m
\end{cases}
\]

(20)

\[
\phi_{m\alpha}^+(x_j, y_j) = \begin{cases} 
\sum_\lambda t_{jm, \lambda\alpha}(E') \sqrt{\frac{\hbar \omega_n}{\mu \lambda}} \psi_{\lambda k'}^+(x_j, y_j) & \text{if } j \neq m \\
\psi_{\alpha k}(x_j, y_j) + \sum_\lambda t_{jj, \lambda\alpha}(E') \sqrt{\frac{\hbar \omega_n}{\mu \lambda}} \psi_{\lambda k'}^+(x_j, y_j) & \text{if } j = m
\end{cases}
\]

(21)
The state at energy $E$ corresponds to an incoming electron in lead $n$, transverse state $\mu$, with longitudinal momentum $p$:

$$\phi_{n\mu}^+(x_i,y_i) = \begin{cases} \sum_\beta t_{in,\beta\mu}(E) \sqrt{\frac{|v_\mu|}{|v_\beta|}} \psi_{\beta\mu}^+(x_i,y_i) & \text{if } i \neq n \\ \psi_{\mu\mu}(x_i,y_i) + \sum_\beta t_{ii,\beta\mu}(E) \sqrt{\frac{|v_\mu|}{|v_\beta|}} \psi_{\beta\mu}^+(x_i,y_i) & \text{if } i = n \end{cases}$$

(22)

$$\phi_{n\mu}^-(x_j,y_j) = \begin{cases} \sum_\delta t_{jj,\delta\mu}(E) \sqrt{\frac{|v_\mu|}{|v_\delta|}} \psi_{\delta\mu}^+(x_j,y_j) & \text{if } j \neq n \\ \psi_{\mu\mu}(x_j,y_j) + \sum_\delta t_{jj,\delta\mu}(E) \sqrt{\frac{|v_\mu|}{|v_\delta|}} \psi_{\delta\mu}^+(x_j,y_j) & \text{if } j = n \end{cases}$$

(23)

Using these states, Equation 19 becomes, setting $\hbar = c = 1$ for simplicity,

$$g_{ij} = \lim_{\omega \to 0} \frac{i}{\omega} \int_{P_i} dt \int_{P_j} dt \sum_{\alpha k n \mu \nu} \bar{J}_i \cdot J_j \cdot Z$$

(24)

where

$$\bar{J}_i = J_{\alpha k,\mu\nu}(x_i,y_i) \delta_{mi} \delta_{ni} + \sum_\beta J_{\alpha k,\beta\mu}(x_i,y_i) t_{in,\beta\mu}(E) \sqrt{\frac{|v_\mu|}{|v_\beta|}} \delta_{ni}$$

$$+ \sum_\nu J_{\nu k,\mu\nu}(x_i,y_i) t_{im,\nu\alpha}(E') \sqrt{\frac{|v_\alpha|}{|v_\nu|}} J_{\nu k,\mu\nu}(x_i,y_i) ,$$

(25)

$$\bar{J}_j = J_{\mu p,\alpha k}(x_j,y_j) \delta_{mj} \delta_{nj} + \sum_\delta J_{\mu p,\lambda k}(x_j,y_j) t_{jm,\lambda\alpha}(E') \sqrt{\frac{|v_\lambda|}{|v_\mu|}} \delta_{nj}$$

$$+ \sum_\lambda J_{\delta p,\alpha k}(x_j,y_j) t_{jn,\delta\mu}(E) \sqrt{\frac{|v_\mu|}{|v_\delta|}} \delta_{mj}$$

$$+ \sum_\delta J_{\delta p,\alpha k}(x_j,y_j) t_{jm,\lambda\alpha}(E') \sqrt{\frac{|v_\lambda|}{|v_\delta|}} J_{\delta p,\lambda k}(x_j,y_j) ,$$

(26)

$$Z = -i \int_0^\infty dt \ e^{i \omega t} [c^{\dagger}_\alpha(t)c_\alpha(t), c^{\dagger}_\mu(0)c_\mu(0)]$$

The fermion creation and annihilation operators refer to scattering wave states. The matrix elements of the current operator are given by:

$$J_{\delta k,\mu\nu}^{\pm\pm}(x_i,y_i) = \frac{\mp e}{2m_i L_i} i(k + p) \chi_{\alpha k}^{\mp\mp}(y_i) e^{\mp ikz_i} \chi_{\mu\nu}^{\pm\pm}(y_i) e^{\pm ipz_i}$$

$$+ \frac{e^2 B}{m_i L_i} y_i \chi_{\alpha k}^{\mp\mp}(y_i) e^{\mp ikz_i} \chi_{\mu\nu}^{\pm\pm}(y_i) e^{\pm ipz_i} ,$$

(27)
\[ J_{\alpha k,up}^\pm \left( x_i, y_i \right) = \frac{\mp e}{2m_i L_i} i\left( k - p \right) \chi_{\alpha k}^\pm \left( y_i \right) e^{\mp ik x_i} \chi_{\mu p}^\mp \left( y_i \right) e^{\mp i p x_i} \]
\[ + \frac{e^2 B}{m L_i} y_i \chi_{\alpha k}^\pm \left( y_i \right) e^{\mp ik x_i} \chi_{\mu p}^\mp \left( y_i \right) e^{\mp i p x_i}, \] (29)

In the traditional language of creation and annihilation operators, an incoming electron-hole pair is created at time 0 in lead \( j \) and the same pair is destroyed as it leaves lead \( i \) at time \( t \). Evaluating \( Z \) for non-interacting electrons,

\[ Z = -i \int_0^\infty dt \ e^{i\omega t} (c_\alpha^\dagger (t) c_\alpha (0) c_\mu (t) c_\mu^\dagger (0) - c_\mu^\dagger (0) c_\mu (t) c_\alpha (0) c_\alpha^\dagger (t)) \] (30)
\[ Z = -i \int_0^\infty dt \ e^{i\left( \omega + i\delta^+ \right)t} (e^{iE't} e^{-iEt}f\left(E'\right)(1 - f\left(E\right)) \]
\[ - e^{-iEt} e^{iE't} f\left(E\right)(1 - f\left(E'\right)) \] (31)
\[ Z = \frac{f\left(E'\right) - f\left(E\right)}{\omega - E + E' + i\delta^+} \] (32)

\( f\left(E\right) \) is the Fermi-Dirac distribution function.

### 2.2.4 Completing the calculation with integration in the complex energy plane

The next step is to convert the sums over \( k \) and \( p \) into integrals over \( E \) and \( E' \),

\[ g_{ij} = \lim_{\omega \to 0} \frac{-L_i L_j}{i\omega (2\pi)^2} \int_{P_i} dy_i \int_{P_j} dy_j \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \sum_{\alpha \mu \alpha' \mu'} \frac{1}{v_\alpha v_\mu} J_i J_j Z \] (33)

From the analytic properties of the \( S \)-matrix elements in the complex energy plane and making use of the fact that \( i \neq j \), one can show that every term in the conductance vanishes except for the one corresponding to an electron incoming in lead \( j \) and outgoing in lead \( i \). For example, the term with current and \( S \)-matrix elements corresponding to an electron incoming in lead \( i \) and outgoing in lead \( j \)

\[ \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \sum_{\alpha \mu \lambda \delta} \frac{J_{\alpha k,up}^\mp \left( x_i, y_i \right) J_{\beta \mu,\lambda \delta \nu}^\pm \left( x_j, y_j \right) t_{ji,\lambda \delta}^\pm \left( E\right) t_{ji,\lambda \delta}^\mp \left( E'\right)}{\sqrt{|v_\lambda||v_\delta||v_\delta||v_\mu|}} \times \frac{f\left(E'\right) - f\left(E\right)}{\omega - E + E' + i\delta^+} \] (34)

vanishes. One can obtain this result by deforming the real-axis \( E \) integral into the lower half plane where \( t_{ji,\delta \mu}^\pm \left( E\right) \) is analytic. For \( E \) in the lower half plane, the physical
values (those with positive real parts) of \( p \) and \( p' \) have negative imaginary parts. Both \( J^-_{ak,\mu p} \) and \( J^{++}_{\delta k',\lambda k''} \), therefore, vanish exponentially for an arbitrarily large semicircular contour in the lower half plane. The real-axis integral consequently vanishes. This result can also be obtained by deforming the real-axis \( E' \) integral into the upper half plane and applying similar arguments.

The only nonvanishing term yields a conductance:

\[
g_{ij} = \lim_{\omega \to 0} \frac{-L_i L_j}{i\omega (2\pi)^2} \int_{P_i} dy_i \int_{P_j} dy_j \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \times \sum_{\alpha \mu \beta \nu} \frac{J^-_{\mu p,ak}(x_i,y_j)J^{++}_{\nu p',\beta k'}(x_i,y_i)t^*_{ij,\alpha \nu}(E')t_{ij,\beta \mu}(E)Z}{\sqrt{|v_{\alpha}||v_{\nu}|v_{\mu}|v_{\beta}|} (35)\]

One can evaluate the integral over \( E' \) by deforming the real-axis contour into the lower half plane where \( t^*_{im,\nu \alpha}(E') \) is analytic, taking note of the pole at \( E' = E - \omega - i0^+ \) as shown in Figure 7. There is no contribution from a large semicircle in the lower half plane because of the exponential vanishing of the current matrix elements. Taking the contribution from the pole and performing the limit,

\[
g_{ij} = \frac{-L_i L_j}{2\pi} \int_{P_i} dy_i \int_{P_j} dy_j \int_{-\infty}^{\infty} dE \sum_{\alpha \mu \beta \nu} J^-_{ak,\mu p}(x_i,y_j)J^{++}_{nk',\beta p'}(x_i,y_i) \left( -\frac{\partial f_{ij,\alpha \nu}(E)}{\partial E} \right) \times \frac{t^*_{ij,\alpha \nu}(E)t_{ij,\beta \mu}(E)}{\sqrt{|v_{\alpha}|v_{\nu}|v_{\mu}|v_{\beta}|} (36)\]

From current conservation, the current matrix elements are constant when integrated over lead cross-sections (that is, independent of \( x_i \) or \( x_j \)) for states of the same energy\([50, 45, 51]\). In order for the oscillatory factors \( e^{i(k-p)x_i}e^{i(p'-k')x_i} \) to be constants, it must be true that \( k = p \) and \( k' = p' \). Because transverse states of the same longitudinal momentum are orthogonal, \( \beta = \nu \) and \( \alpha = \mu \). Using Equations 13 and 28 in Equation 36 yields the final result,

\[
g_{ij} = \frac{e^2}{2\pi h} \sum_{\alpha \beta} \int_{-\infty}^{\infty} |t_{ij,\beta \alpha}(E,B)|^2 \left( -\frac{\partial f_{ij,\alpha \nu}(E,B)}{\partial E} \right) dE (37)\]

where \( h \) is now explicitly included to yield units of conductance. Introducing Büttiker’s transmission coefficient \( T_{ij} \),

\[
g_{ij} = \frac{e^2}{h} \int_{-\infty}^{\infty} T_{ij}(E,B) \left( -\frac{\partial f}{\partial E} \right) dE (38)\]
Figure 7: Contour integral used to evaluate Equation 35. $t_{im,\nu}\alpha(E')$ is analytic everywhere in the lower half plane. $g_{ij}$ picks up a contribution from the pole at $E' = E - \omega - i0^+$. 
where \( i \neq j \). *Velocity factors in the numerator from the current matrix elements cancel velocity factors in the denominator coming from the one-dimensional density of states.* It is because of this cancellation that the current in a probe is shared equally among the occupied modes at the Fermi energy. This is a very important aspect of quasi-one-dimensional transport which holds in the limit of linear response. These conductance coefficients lead to the multiprobe Landauer-Büttiker formula (Equation 5) at zero temperature[45, 51].

### 2.3 Four-terminal resistances

Equation 38 gives the conductance of the multiprobe system; that is, the current response to voltages applied to each of the leads. In Chapter 6, we describe a technique for extracting these conductances and, therefore, the transmission coefficients directly. In experiments, however, one frequently measures resistances in which fixed currents are applied to some of the leads and voltages are measured between others. It is possible to write general expressions for four-terminal resistances in terms of transmission coefficients[44].

Equation 5 in matrix form is:

\[
\frac{h}{2e^2} \bar{I} = T' \bar{V} \tag{39}
\]

where \( T' \) is a \( N_L \times N_L \) matrix with elements \( T'_{ij} = T_{ij} - N_i \delta_{ij} \). \( N_L \) is the number of leads. The desired resistances are given by:

\[
R_{kl,mn} = \frac{V_m - V_n}{I} \left\{ \begin{array}{ll}
I = I_k = -I_l \\
I_i = 0 \text{ for } i \neq k, l
\end{array} \right. \tag{40}
\]

The requirements of current conservation and the arbitrariness of the zero voltage reference imply that each row and column of \( T' \) must sum to zero. Therefore, all the subdeterminants, \( D \), of \( T' \) are equal and more importantly \( T' \) is itself singular; that is, \( \det(T') = 0 \). One can, therefore, not find the voltages in Equation 39 by simply inverting \( T' \)[45].
CHAPTER 2. LANDAUER-BÜTTIKER FORMULA

The trick is to shift the voltage reference and write Equation 39 as:

\[
\begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
\vdots
\end{pmatrix} = \frac{2e^2}{h} \begin{pmatrix}
T_{11} - N_1 & T_{12} & T_{13} & T_{14} & \\
T_{21} & T_{22} - N_2 & T_{23} & T_{24} & \\
T_{31} & T_{32} & T_{33} - N_3 & T_{34} & \\
T_{41} & T_{42} & T_{43} & T_{44} - N_4 & \\
\vdots & \vdots & \vdots & \vdots & \\
\end{pmatrix} \begin{pmatrix}
0 \\
V_2 - V_1 \\
V_3 - V_1 \\
V_4 - V_1 \\
\vdots
\end{pmatrix}
\]

(41)

The \(N_L \times N_L\) matrix equation reduces to a \((N_L - 1) \times (N_L - 1)\) matrix equation which is now invertible.

\[
\begin{pmatrix}
I_2 \\
I_3 \\
I_4 \\
\vdots
\end{pmatrix} = \frac{2e^2}{h} \begin{pmatrix}
T_{22} - N_2 & T_{23} & T_{24} & \\
T_{32} & T_{33} - N_3 & T_{34} & \\
T_{42} & T_{43} & T_{44} - N_4 & \\
\vdots & \vdots & \vdots & \\
\end{pmatrix} \begin{pmatrix}
V_2 - V_1 \\
V_3 - V_1 \\
V_4 - V_1 \\
\vdots
\end{pmatrix}
\]

(42)

In general, one solves this matrix equation and uses Equation 40 to find \(R_{kl, mn}\). For the case of the four-probe conductor, one can derive, by application of Cramer’s rule\[62\] to the \(3 \times 3\) matrix equation of the form given in Equation 42, a very simple expression for the four-terminal resistances. Inverting the matrix,

\[
\begin{pmatrix}
V_2 - V_1 \\
V_3 - V_1 \\
V_4 - V_1
\end{pmatrix} = \frac{h}{2e^2 D} A \begin{pmatrix}
I_2 \\
I_3 \\
I_4
\end{pmatrix}
\]

(43)

where

\[
A_{11} = (T_{33} - N_3)(T_{44} - N_4) - T_{34}T_{43}
\]

(44)

\[
A_{21} = T_{34}T_{42} - T_{32}(T_{44} - N_4)
\]

(45)

\[
A_{31} = T_{32}T_{43} - T_{42}(T_{33} - N_3)
\]

(46)

\[
A_{12} = T_{24}T_{43} - T_{23}(T_{44} - N_4)
\]

(47)

\[
A_{22} = (T_{22} - N_2)(T_{44} - N_4) - T_{42}T_{24}
\]

(48)

\[
A_{32} = T_{23}T_{42} - (T_{22} - N_2)T_{43}
\]

(49)

\[
A_{13} = T_{23}T_{34} - T_{24}(T_{33} - N_3)
\]

(50)
\[ A_{23} = T_{24}T_{32} - (T_{22} - N_2)T_{34} \] (51)
\[ A_{33} = (T_{22} - N_2)(T_{33} - N_3) - T_{23}T_{32} \] (52)

and \( D \) is the determinant of the \( 3 \times 3 \) matrix in Equation 42 (or any subdeterminant of the \( 4 \times 4 \) matrix in Equation 39). Computing, for example, \( R_{24,13} \), \( I_2 = I \), \( I_4 = -I \), and \( I_1 = I_3 = 0 \):
\[ R_{24,13} = \frac{\hbar}{2e^2} \frac{V_3 - V_1}{I} = \frac{\hbar}{2e^2} \frac{T_{34}T_{42} - T_{32}(T_{44} - N_4) - [T_{24}T_{32} - (T_{22} - N_2)T_{34}]}{D} \] (53)

but
\[ T_{22} - N_2 = T_{12} - T_{32} - T_{42} \] (54)

and
\[ T_{44} - N_4 = T_{14} - T_{24} - T_{34} \] (55)

Therefore, one finds,
\[ R_{24,13} = \frac{\hbar}{2e^2} \frac{T_{12}T_{34} - T_{14}T_{32}}{D} \] (56)
or, in general[44],
\[ R_{k,lm} = \frac{\hbar}{2e^2} \frac{T_{mk}T_{nl} - T_{ml}T_{nk}}{D} \] (57)

### 2.4 High-field transport

Application of the Landauer-Büttiker formula to quantum diffusive and ballistic transport requires calculation of the transmission coefficients. In a fully quantum mechanical case, this involves solving the Schrödinger equation. A numerical technique for doing this convenient to multiprobe structures is discussed in Chapter 3 and has been used to study transport in the quantum diffusive regime[63]. In the classical limit, the transmission coefficients can be evaluated for classical ballistic particles[64]. Using a classical current injection distribution\(^5\), a Monte Carlo evaluation of the transmission coefficients involves tracing ballistic trajectories and counting how many electrons emerge from each lead.

\(^5\)For the case of hard-wall potentials and zero magnetic field, this is given by \( P(\alpha) = \cos(\alpha)/2 \). For a general formulation, see Baranger[65].
In the integral quantum Hall effect regime, calculation of the transmission coefficients becomes trivial\cite{66, 67, 68, 69, 70}, and the Landauer-Büttiker formula can become a powerful means of formulating transport. As indicated in Section 1.2.3, this formulation is essential when a two-dimensional local resistivity tensor can no longer be defined in samples in which one or more dimensions becomes comparable with the inelastic scattering length. It is, however, a completely general approach that is also successful in describing high-field transport on macroscopic length scales, as mentioned in Section 2.1.

The multiprobe, multichannel Landauer-Büttiker formula relates conductance to the transmission and reflection of Fermi-level states. In the integer quantum Hall effect regime, this leads to an edge-state picture, first proposed by Halperin\cite{71}. Consider the generalized sample shown in Figure 5 in the presence of a uniform magnetic field oriented such that the Lorentz force steers current in a clockwise direction. Electrons in the leads are pushed to either one edge or the other, depending on the direction of their longitudinal velocity. These edge states persist in the scattering region even in the presence of disorder as long as the magnetic field is sufficiently strong and the Fermi energy is far from a bulk Landau level. In this simplest case, \( T_{n+1,n} = N_{LL} \), where \( N_{LL} \) is the number of Landau levels, which is assumed to be the same in each lead and in the scattering region. All other transmission coefficients are zero. In particular, there is no reflection; \( T_{nn} = 0 \). The \( T' \) matrix introduced in Equation 39 becomes, for example, for the six-lead case,

\[
T' = \begin{pmatrix}
-N_{LL} & 0 & 0 & 0 & 0 & N_{LL} \\
N_{LL} & -N_{LL} & 0 & 0 & 0 & 0 \\
0 & N_{LL} & -N_{LL} & 0 & 0 & 0 \\
0 & 0 & N_{LL} & -N_{LL} & 0 & 0 \\
0 & 0 & 0 & N_{LL} & -N_{LL} & 0 \\
0 & 0 & 0 & 0 & N_{LL} & -N_{LL}
\end{pmatrix}
\]  

(58)

If two probes are used to carry the current, then it is easy to show, using the methods of Section 2.3, that the resistance between two voltage probes on either side of the current path is \( R_H = \frac{h}{2e^2 N_{LL}} \) (cf. Equation 3), the quantized Hall resistance. On the other hand, two voltage probes on the same side of the current path yield a vanishing
longitudinal resistance, \( R_L = 0 \) (cf. Equation 4). The suppression of backscatter, \( T_{nn} = 0 \), is a necessary condition for quantum Hall transport. Between plateaus, the reflection coefficients, \( T_{nn} \), do not vanish and \( R_L \) is finite. The exact criterion for being in the quantum Hall regime in the presence of disorder for the case of the quantum wire is discussed in Chapter 4.

The power of this approach also lies in its application to more complex phenomena, such as the breakdown of the quantum Hall effect due to a scattering path [72], the anomalous integer quantum Hall effect[26], and backscatter experiments in gated quantum Hall systems[10, 66]. To demonstrate this, let us consider an experiment in which there is a single localized region of breakdown, as shown in Figure 8. If this

![Figure 8: Six-probe conductor with a single localized region of breakdown. Three edge states are occupied at the Fermi energy throughout the structure, except through the localized constriction, which allows only a single mode of transport \( T = 1 \).](image-url)
region admits only a transmission $T$, then the matrix $T'$ becomes:

$$
T' = \begin{pmatrix}
-N_{LL} & 0 & N_{LL} - T & 0 & 0 & T \\
N_{LL} & -N_{LL} & 0 & 0 & 0 & 0 \\
0 & N_{LL} & -N_{LL} & 0 & 0 & 0 \\
0 & 0 & T & -N_{LL} & 0 & N_{LL} - T \\
0 & 0 & 0 & N_{LL} & -N_{LL} & 0 \\
0 & 0 & 0 & 0 & N_{LL} & -N_{LL}
\end{pmatrix}
$$

(59)

For $I_2 = I, I_5 = -I, I_1 = I_3 = I_4 = I_6 = 0$, solving the matrix equation, one finds:

$$
R_{25,31} = R_{25,46} = \frac{h}{2e^2} \frac{1}{N_{LL}}
$$

(60)

$$
R_{25,34} = R_{25,16} = \frac{h}{2e^2} \left( \frac{1}{T} - \frac{1}{N_{LL}} \right)
$$

(61)

$$
R_{25,36} = \frac{h}{2e^2} \frac{1}{T}
$$

(62)

$$
R_{25,14} = \frac{h}{2e^2} \left( \frac{1}{T} - \frac{2}{N_{LL}} \right)
$$

(63)

Equation 62 is simply the two-terminal resistance of the localized breakdown region and, in fact, forms the basis for using multiprobe measurements to extract two-terminal resistances and conductances in the presence of magnetic fields (see Section 5.3.2 and Section 6.2.1). This technique easily generalizes to multiple regions of breakdown and experiments in which probes inject different numbers of Landau levels.

### 2.5 Summary

This chapter reviews the use of the Landauer-Büttiker formalism to describe transport in mesoscopic systems and presents a general method for relating four-terminal resistances to the transmission coefficients. This chapter has also outlined a linear response derivation of the Landauer-Büttiker formula which uses the analytic properties of the microscopic multichannel, multiprobe $S$-matrix.
Chapter 3

Recursive Green’s Function Technique

3.1 Introduction

The transmission coefficients $T_{ij}$ for coherent transport introduced in Equation 5 are obtained in general by solving the Schrödinger equation within the two-dimensional mesoscopic conductor. The recursive Green’s function technique implemented in this chapter is a particularly powerful method for performing this computation numerically for arbitrary two-probe geometries, which allows the incorporation of local site disorder (that is, short-ranged elastic scattering centers) and magnetic fields. This is the first complete exposition[73] of the technique to high magnetic fields[74, 75] built upon several references in the literature[76, 77, 78, 79, 51]. The technique can also be extended to model transmission in multiprobe conductors[79]. These calculations are used in Chapter 4 to study the transition from diffusive to quantum Hall transport in narrow wires. This involves a study of the suppression of backscatter in disordered wires as a magnetic field is applied. The recursive Green’s function formalism outlined here is also used to compute the momentum distribution of electrons emitted from a quantum point contact in Chapter 6 at zero magnetic field.
CHAPTER 3. RECURSIVE GREEN’S FUNCTION TECHNIQUE

3.2 Developing the formalism

3.2.1 Discretizing the free-electron Hamiltonian

The two-dimensional free-electron Hamiltonian in the presence of a magnetic field is given by:

$$\mathcal{H} = \frac{1}{2m} \left( \vec{p} - \frac{e}{c} \vec{A} \right)^2 + V(x, y)$$  \hspace{1cm} (64)

where $V(x, y)$ is a local potential. Since computations are to be done numerically, a natural starting point is to discretize the Hamiltonian. Free space is broken up into a two-dimensional grid of equally-spaced lattice points indexed by $n$ and $m$. Operators act on states $|n, m\rangle$ localized on these sites.

Discretizing the canonical momenta:

$$\left( \vec{p} - \frac{e}{c} \vec{A} \right)_x = -\frac{i\hbar}{2a} \sum_{n,m} \left[ e^{\frac{i\varepsilon A_y}{\hbar c}} |n, m > n + 1, m\rangle - e^{\frac{-i\varepsilon A_y}{\hbar c}} |n + 1, m > n, m\rangle \right]$$  \hspace{1cm} (65)

$$\left( \vec{p} - \frac{e}{c} \vec{A} \right)_y = -\frac{i\hbar}{2a} \sum_{n,m} \left[ e^{\frac{i\varepsilon A_x}{\hbar c}} |n, m > n, m + 1\rangle - e^{\frac{-i\varepsilon A_x}{\hbar c}} |n, m + 1 > n, m\rangle \right]$$  \hspace{1cm} (66)

where $a$ is the lattice constant.

Using these expressions in Equation 64, the free-electron Hamiltonian assumes an Anderson tight-binding form:

$$\mathcal{H} = \frac{\hbar^2}{2ma^2} \sum_{n,m} \left[ -e^{\frac{i\varepsilon A_y}{\hbar c}} |n, m > n + 1, m\rangle - e^{\frac{-i\varepsilon A_y}{\hbar c}} |n + 1, m > n, m\rangle \right.$$  

$$+ 4 |n, m > n, m\rangle - e^{\frac{i\varepsilon A_x}{\hbar c}} |n, m > n, m + 1\rangle$$  

$$- e^{\frac{-i\varepsilon A_x}{\hbar c}} |n, m + 1 > n, m\rangle + V_{n,m} |n, m > n, m\rangle \right]$$  \hspace{1cm} (67)

where one introduces a site energy $V_{n,m}$ with units of $\frac{\hbar^2}{2ma^2}$. The $V_{n,m}$ can be used to model any local potential. Using dimensionless units with $[E] = \frac{\hbar^2}{2ma^2}$, $[B] = \frac{\hbar c}{ea^2} = \frac{A_0}{a}$, and $[k] = \frac{1}{a}$ and choosing Landau gauge as in Chapter 2, $\vec{A} = (-yB, 0)$, the Hamiltonian becomes:

$$\mathcal{H} = \sum_{n,m} \left\{ -e^{-iBm} |n, m > n + 1, m\rangle - e^{iBm} |n + 1, m > n, m\rangle + 4 |n, m > n, m\rangle \right.$$  

$$- |n, m > n, m + 1\rangle - |n, m + 1 > n, m\rangle + V_{n,m} |n, m > n, m\rangle \right\}$$  \hspace{1cm} (68)
The nearest-neighbor site interactions are depicted in Figure 9.

![Diagram showing nearest-neighbor site interactions in the Anderson tight-binding model in Landau gauge.]

Figure 9: Nearest-neighbor site interactions in the Anderson tight-binding model in Landau gauge.

### 3.2.2 Properties of the Green's function

The recursive Green's function technique was originally developed to perform calculations on a purely one-dimensional chain system[76]. By replacing each site on the linear chain by an entire column of sites, the technique is easily extended to model structures with large aspect ratios; that is, structures which are longer than they are wide[77]. In what follows bold-faced **CAPITAL** letters denote matrices in this space of transverse lattice sites. Indices denote the columns. The tight-binding Hamiltonian, therefore, has the general form:

\[ \mathcal{H} = \sum_{ij} H_{ij} |i \rangle \langle j| \]  

(69)

In order to determine how an electron is transmitted through the structure, one computes the Green's function \( G_{ij} \), the propagator from column \( i \) to column \( j \). This
matrix satisfies the relation:

\[(EI - H_{ij})G_{ij} - H_{ii+1}G_{i+1j} - H_{ii-1}G_{i-1j} = I\delta_{ij}\]  

(70)

where \(I\) is the identity matrix and \(E\) is the energy, which in general includes a small imaginary part, \(E \rightarrow E + i\delta\). This ensures irreversibility and provides a mechanism for introducing inelastic scattering processes which are "continuously" distributed through the structure. The Green's functions for a system \(N\) columns long can be related to the Green's functions for a system \(N+1\) columns long by Dyson's equation:

\[G_{ij}^{(N+1)} = G_{ij}^{(N)} + G_{iN}^{(N)}H_{NN+1}G_{N+1j}^{(N+1)}\]  

(71)

One can set \(\delta = 0\) and achieve irreversibility and dissipation by making the structure infinitely long by the attachment of semi-infinite ideal leads. Once an electron is outgoing in a lead, it never returns. This is equivalent to thermalizing with a reservoir and is the same theoretical construction used in the derivation of Section 2.2. Figure 10 shows the replacement of the reservoirs with semi-infinite ideal leads and the discretization of the resulting two-port waveguide structure. Hard-wall potentials define the sides of the device. The recursive Green's function technique is used to compute the transmission coefficients between planes \(i\) and \(j\) within the asymptotic regions of the ideal leads.

### 3.2.3 Recursion relations

Several important recursion relations, necessary for the calculation, can be derived from Equations 70 and 71. Beginning from the left, one can calculate the Green's function \(G_{jj}^L\) for a system in which all columns to the right of \(j\) are deleted. One can add successive columns to the structure to the right. From Equation 71,

\[G_{jj+1}^L = G_{jj}^L V_x G_{j+1j+1}^L\]  

(72)

From Equation 70,

\[G_{jj}^{-1} G_{j+1j+1}^L - V_x^\dagger G_{jj+1}^L = I\]  

(73)
Figure 10: Discretization of a two-port waveguide structure defined by hard-wall potentials. (a) Two-port waveguide with reservoirs. (b) Reservoirs modelled with semi-infinite ideal leads. (c) Discretization of the structure in (b).
where \( V_x = H_{i+1} \) defines the connection between successive columns. With the choice of Landau gauge, this matrix is given by:

\[
V_x = \begin{pmatrix}
-e^{-iB} & 0 & 0 & 0 \\
0 & -e^{-2iB} & 0 & 0 \\
0 & 0 & -e^{-3iB} & 0 \\
0 & 0 & 0 & -e^{-4iB} \\
\vdots
\end{pmatrix}
\]  

(74)

It is immediately evident that \( V_x^{-1} = V_x^* = V_x^\dagger \). \( G_{jj}^o \) is the Green’s function of the isolated \( j \) column and is given by:

\[
G_{jj}^{o-1} = E \mathbf{I} - H_{jj}
\]

(75)

where \( H_{jj} \) is the Hamiltonian matrix for the isolated column:

\[
H_{jj} = \begin{pmatrix}
V_{j1} + 4 & -1 & 0 & 0 \\
-1 & V_{j2} + 4 & -1 & 0 \\
0 & -1 & V_{j3} + 4 & -1 \\
0 & 0 & -1 & V_{j4} + 4 \\
\vdots
\end{pmatrix}
\]

(76)

\( E \) is the energy. From Equation 72,

\[
V_x^\dagger G_{j-1j}^L = V_x^\dagger G_{j-1j-1}^L V_x G_{jj}^L
\]

(77)

From Equations 77 and 73,

\[
G_{jj}^{o-1} G_{jj}^L - \mathbf{I} = V_x^\dagger G_{j-1j-1}^L V_x G_{jj}^L
\]

(78)

This gives the desired recursion relation:

\[
G_{jj}^L = [G_{jj}^{o-1} - V_x^\dagger G_{j-1j-1}^L V_x]^{-1}
\]

(79)

Beginning from the right, one can similarly build up the Green’s function in which all columns to the left of \( j \) are deleted:

\[
G_{jj}^R = [G_{jj}^{o-1} - V_x G_{j+1j+1}^R V_x^\dagger]^{-1}
\]

(80)
CHAPTER 3. RECURSIVE GREEN’S FUNCTION TECHNIQUE

There are three additional recursion relations which allow one to build up the Green’s function between any two columns. They all follow simply from Equations 70 and 71.

\[
G_{jj} = [G_{jj}^{o} - V_x G_{j+1j+1}^{R} V_x^\dagger - V_x^\dagger G_{j-1j-1}^{L} V_x]^{-1}
\]  

(81)

\[
G_{jj'+1} = G_{jj'} V_x G_{j+1j'+1}^{R}
\]

(82)

\[
G_{j-1j'} = G_{j-1j-1}^{L} V_x^\dagger G_{jj'}
\]

(83)

3.2.4 $G_{jj}^L$ and $G_{jj}^R$ for the semi-infinite ideal lead

One must now compute $\tilde{G}_{jj}^L$ and $\tilde{G}_{jj}^R$, $G_{jj}^L$ and $G_{jj}^R$ for the semi-infinite ideal lead. These must satisfy the relations:

\[
\tilde{G}_{jj}^L = [G_{jj}^{o} - V_x^\dagger \tilde{G}_{jj}^L V_x]^{-1}
\]

(84)

\[
\tilde{G}_{jj}^R = [G_{jj}^{o} - V_x \tilde{G}_{jj}^R V_x^\dagger]^{-1}
\]

(85)

The required forms are:

\[
\tilde{G}_{jj}^L = -\frac{1}{2} \sum_\alpha \left[ V_x \chi_{ak}^- \chi_{ak}^- \dagger e^{-ik} + e^{-ik} \chi_{ak}^+ \chi_{ak}^+ \dagger V_x^\dagger \right]
\]

(86)

and

\[
\tilde{G}_{jj}^R = -\frac{1}{2} \sum_\alpha \left[ V_x^\dagger \chi_{ak}^+ \chi_{ak}^+ \dagger e^{-ik} + e^{-ik} \chi_{ak}^- \chi_{ak}^- \dagger V_x \right]
\]

(87)

where the sums are over Fermi-level states with longitudinal momentum $k > 0$. $\chi_{ak}^+$ is the transverse lead eigenstate of mode $\alpha$ for an outgoing electron (a vector in the space of transverse lattice sites). $\chi_{ak}^-$ is the transverse lead eigenstate of mode $\alpha$ for an incoming electron. For $B = 0$, these forms simplify to:

\[
\tilde{G}_{jj}^R = \tilde{G}_{jj}^L = -\sum_\alpha e^{-ik} \chi_{\alpha} \chi_{\alpha}^\dagger
\]

(88)

$\chi_{\alpha}$ is now a transverse lead eigenstate independent of longitudinal momentum.
### 3.2.5 Calculating transmission coefficients from the Green’s function

Another important relationship is required, the one connecting the Green’s function to the transmission coefficients. Given the propagator $G_{ij}$ from column $i$ to column $j$ in the asymptotic region of lead $i$ to column $j$ in the asymptotic region of lead $j$, it is straightforward to calculate the transmission coefficients. Following Baranger and Stone[51], one can define a velocity operator:

$$ J_n = \frac{1}{2i} \sum_m \left\{ e^{-iB_m}|n, m\rangle < n + 1, m\rangle - e^{iB_m}|n + 1, m\rangle < n, m\rangle \right\} \quad (89) $$

Writing this in matrix form:

$$ J_n = \begin{pmatrix}
\sin(k - B) & 0 & 0 \\
0 & \sin(k - 2B) & 0 \\
0 & 0 & \sin(k - 3B)
\end{pmatrix} \quad (90) $$

The velocity operator has the property:

$$ \chi_{\alpha k}^\dagger J_n \chi_{\beta k'}^\pm = \pm |v_{\alpha k}| \delta_{\alpha \beta} \quad (91) $$

where $v_{\alpha k}$ is the longitudinal velocity of transverse state $\chi_{\alpha k}^\pm$. $\chi_{\alpha k}^\pm$ and $\chi_{\beta k'}^\pm$ are, of course, transverse eigenstates of the same energy. Then[51]

$$ T_{ij, mn} = |t_{ij, mn}|^2 = |\chi_{nk}^+ J_i G_{ij} J_j \chi_{nk'}^-|^2 / v_{mk} v_{nk'} $$

Transverse state $\chi_{mk}^+$ is outgoing in lead $i$, while the transverse state $\chi_{nk'}^-$ is incoming in lead $j$. For the case of $B = 0$, this assumes a simpler form:

$$ T_{ij, mn} = |t_{ij, mn}|^2 = |\chi_{m}^+ G_{ij} \chi_{n}|^2 v_m v_n \quad (93) $$

The transverse state $\chi_{m}$, now independent of longitudinal momentum, is outgoing in lead $i$, while the transverse state $\chi_{n}$ is incoming in lead $j$. 
3.3 Two-probe recursive Green’s function algorithm

Figure 11 is a flow diagram for the two-probe recursive Green’s function algorithm. Figure 12 shows a schematic implementation of the procedure for a particular model potential. The first step is to calculate the transverse eigenstates and the longitudinal wave vector at the Fermi energy for the ideal lead by diagonalizing the matrix:

\[
\begin{pmatrix}
4 - 2 \cos(k - B) & -1 & 0 & 0 \\
-1 & 4 - 2 \cos(k - 2B) & -1 & 0 \\
0 & -1 & 4 - 2 \cos(k - 3B) & -1 \\
0 & 0 & -1 & 4 - 2 \cos(k - 4B) \\
\vdots & & & \ddots
\end{pmatrix}
\]  

(94)

which is the Hamiltonian matrix in the space of transverse indices for ideal wire states with longitudinal momenta \(k\). Using these eigenstates and wave vectors, \(G_{ii}^L\) is computed for the semi-infinite ideal lead to the left of the structure by using Equation 86 (See Figures 11a and 12a). One can then build up \(G_{ii}^L\) to the column \(i - 1\) by successive application of Equation 79 (See Figures 11b and 12b). \(G_{ii}^R\) for the right-hand is computed in the same manner used to compute \(G_{ii}^L\). One similarly builds \(G_{ii}^R\) up to the column \(i + 1\) by successive application of Equation 80 (See Figures 11c and 12c). In this case, the values of \(G_{ii}^R\) are stored for each column between \(i + 1\) and \(j\) (See Figures 11d and 12d). \(G_{i+i-1+i+1}^R\) and \(G_{i-i-1}^L\) are connected to compute \(G_{ii}\) using Equation 81 (See Figures 11e and 12e). It is now straightforward to compute \(G_{ij}\) from \(G_{ii}\) and the stored values of \(G_{ij}\) for \(i + 1\) to \(j\) by repeated application of Equation 81 (See Figures 11f and 12f). The \(T_{ij}\) are then extracted from \(G_{ij}\) by means of Equation 92.

In the example of Figure 12, a wire five constants wide is attached to a wire ten lattice constants wide. The simplest implementation uses all \(10 \times 10\) matrices. When the wire is only five lattice constants wide, diagonal elements can be set to large negative values to exclude the electron. Computer implementation of the technique is rather straightforward. Unlike a transfer matrix approach, this technique is inherently
CHAPTER 3. RECURSIVE GREEN'S FUNCTION TECHNIQUE

Calculate the transverse eigenstates and longitudinal Fermi wavevectors for the ideal lead

\[ \rightarrow \]

a. Compute \( G^L \) for the semi-infinite ideal lead

\[ \rightarrow \]

b. Compute \( G^L \) up to \( i-1 \) by building up the Green's function one column at a time using the recursion relation

\[ G^L_{jj} = [G^o_{jj} - V_x G^L_{j-i-1} V_x]^{-1} \]

\[ \rightarrow \]

c. Compute \( G^R \) for the semi-infinite ideal lead

\[ \rightarrow \]

d. Compute \( G^R \) up to \( i+1 \) by building up the Green's function one column at a time using the recursion relation

\[ G^R_{jj} = [G^o_{jj} - V_x G^R_{j+i+1} V_x]^{-1} \]

Save \( G^R_{ii} \) for each column between \( i+1 \) and \( j \)

\[ \rightarrow \]

e. Compute \( G_{ii} \) using

\[ G_{jj} = [G^o_{jj} - V_x G^R_{j+i+1} V_x - V_x G^L_{j-i-1} V_x]^{-1} \]

\[ \rightarrow \]

f. Compute \( G_{ij} \) using the values of \( G_R \) stored for \( i+1 \) to \( j \) and repeatedly applying:

\[ G_{jj'+1} = G_{jj'} V_x G^R_{j'+1} \]

\[ \rightarrow \]

Use \( G_{ij} \) to compute the transmission coefficients

Figure 11: Flow diagram for the two-probe recursive Green's function algorithm.
Figure 12: Implementing the two-probe recursive Green's function calculation. In this case, the geometry is a wire five lattice constants wide attached to a wire ten lattice constants wide. Labels a – f refer to the flow chart in Figure 11.
very numerically stable. Each step in the recursion to compute $G^R$ through the structure requires a matrix inversion, which quickly limits the width of the structure that can be simulated in a reasonable amount of computer time.

3.4 Summary

This chapter outlines the recursive Green's function technique for numerically computing the transmission coefficients of a phase-coherent two-probe mesoscopic conductor. The algorithm presented is general to arbitrary local potentials and magnetic fields.
Chapter 4

Quantum Hall Effect in Quantum Wires

4.1 Introduction

The suppression of backscatter is a necessary condition for quantum Hall transport [22] in an edge-state picture, as demonstrated in Section 2.4. This chapter considers the transition from the quantum diffusive to the integral quantum Hall effect regimes. The suppression of backscattering is studied theoretically by an exact calculation of the two-terminal conductance of a phase-coherent disordered quasi-one-dimensional quantum wire as a magnetic field is applied[73]. This allows an accurate determination of the requirements for edge-state formation and high-field transport in quasi-one-dimensional systems. These calculations use the high-field extension of the recursive Green’s function technique presented in Chapter 3. Since this algorithm is for the two-probe case, the discussion is limited to the quantum wire. The results can, however, be easily generalized to an arbitrary multiprobe geometry.

The conductance quantization in split-gate ballistic wires[18, 19] has been studied experimentally in the presence of magnetic fields[80]. The observed magnetic depopulation of the subbands in the quasi-one-dimensional channel[81] is explicitly interpreted as the transition[82] between zero-field conductance quantization and the two-terminal quantum Hall effect[83] in the ballistic limit.
CHAPTER 4. QUANTUM HALL EFFECT IN QUANTUM WIRES

It has been known for some time that in quantum wires with significant disorder, a negative magnetoresistance about zero field[84] is generally observed. This is associated with scattering within the wires rather than at the sidewalls[85]. It has already been suggested by van Houten, et al.[67] that this is a manifestation of reduced backscatter in the presence of a magnetic field. At high enough fields, one enters the quantum Hall regime when backscatter is completely suppressed[22]. Recent experiments by van Wees et al.[86] have confirmed that conductance fluctuations in disordered quantum wires evolve into a quantized two-terminal resistance with the application of sufficiently large magnetic fields. This chapter presents numerical calculations clarifying this experimental result. This work complements recent studies by Ando showing the strong energy dependence of the inverse localization length of edge states in disordered quantum wires[87].

4.2 Model system

The calculation is done on a lattice $W$ lattice constants wide as shown in Figure 13. The disordered region consists of a zone $L$ lattice constants long where random site energies are chosen to have a rectangular distribution between $V$ and $-V$. These site energies are used to model short-ranged elastic scattering centers in the channel, such as those produced by screened ionized impurities or heterointerface roughness[88, 89]. Hard-wall boundary conditions are chosen in the transverse direction. Semi-infinite leads are attached at the ends of the wire. In this chapter, spin splitting is ignored and spin degeneracy is assumed since spin degrees of freedom are not important to the results presented here. The issue of spin splitting in quantum wires is discussed in detail in Chapter 5. The conductance is computed from the multichannel Landauer-Büttiker formula[22] $g = \frac{2e^2}{h} Tr(t^\dagger t)$ with transmission matrices determined at planes far enough away from the disordered region that evanescent modes have decayed away. Normalized units as introduced in Chapter 3 are used. Specifically, $[E] = \frac{h}{2ma^2}$, $[k] = \frac{1}{a}$, and $[B] = \frac{2a^2}{e^2}$. The Green's function needed to obtain the transmission coefficients is calculated by means of a recursive Green's function technique, as discussed in Chapter 3.
Figure 13: Model geometry for the wire $W$ lattice sites wide with a disordered region $L$ lattice sites long. Transmission coefficients are computed at planes $i$ and $i'$ far enough away from the disordered region that evanescent modes have decayed away.

4.3 Edge states in the quantum wire

From the scaling theory of high-field transport in two-dimensional electron gases\cite{90, 91}, the electron localization length $\xi$ for a given bulk Landau level follows a power law near some singular energy $E_c$,

$$\xi = \xi_0 (E - E_c)^{-\nu}$$  \hspace{1cm} (95)

where $\nu$ is some universal critical exponent. $E_c$ is the energy at which bulk conduction occurs even at $T = 0$ and is given by the corresponding Landau-level energy. Furthermore, this critical energy is known to levitate, or float, in the presence of increasing disorder according to\cite{92}:

$$E_c = (n + 1/2) \hbar \omega_c \left[ 1 + \frac{1}{(\omega_c \tau)^2} \right]$$  \hspace{1cm} (96)

where $n$ is the spin-degenerate Landau-level index. $\tau$ is the relaxation time associated with the disorder. In the case of a wide two-dimensional electron gas, the sample is in the quantum Hall regime when the Fermi energy lies in the mobility gap between
Landau levels; that is, when the localization length for states at the Fermi level is much smaller than the inelastic mean free path $L_{in}$. Extended bulk states carry the Hall current.

In the case of quasi-one-dimensional systems, it is useful to employ a Landauer-Büttiker picture for quantum Hall transport in which only states at the Fermi energy carry current, as discussed in Section 2.4. The edges now become very important, and current-carrying states in the quantum Hall regime are confined to the steep potential profiles found at these edges. Figure 14 shows the energy dispersion curves ($E$ versus $k$) for a ballistic wire 20 lattice constants wide for magnetic fields of 0, 0.1, and 0.2. As the magnetic field is applied, the center of each curve flattens out to form a very high density of zero-velocity bulk Landau-level states. The zero-velocity bulk Landau-level states have a spatial extent given by $l = l_B(n + 1/2)^{1/2}$ where $l_B = B^{-1/2}$, the magnetic length in normalized units. The states with finite velocity become spatially localized at the edges of the wire for nonzero magnetic fields. To show this, edge-state properties for the same ballistic wire are computed for magnetic fields of 0.1 and 0.2. Figure 15 shows the distance of the edge-state wavefunctions ($< y >$) from the center and the wavefunction extents ($(< y^2 > - < y >^2)^{1/2}$) as a function of Fermi wave vector in units of $\pi/W$.

In the presence of disorder, edge states hybridize with bulk Landau-level states. In this quasi-one-dimensional case, one must identify two localization lengths, one along the length of the wire ($L_{loc}$) and one that describes the transverse spatial extent of these hybrid states ($l_{loc}$). $l_{loc}$ will be greater than the ballistic wavefunction extent computed in the absence of disorder in Figure 15 and can be expected to increase with increasing disorder and for Fermi energies which closely approach the high density of states at the bulk Landau-level energies[23]. In the quantum Hall regime, both elastic and inelastic backscatter is suppressed. This requires $l_{loc}$ be much less than the wire width so that there is negligible overlap between electron edge states at opposite sides of the wire. Fermi level edge states become extended longitudinally on the scale of the inelastic scattering length$^1$ ($L_{loc} \gg L_{in}$)[87]. When the Fermi

$^1$If the sample length $L$ is less than $L_{in}$, then the condition for being extended longitudinally becomes $L_{loc} \gg L$. 
Figure 14: Energy dispersion curves ($E$ versus $k$) for a ballistic wire 20 lattice constants wide for magnetic fields of (a) $B = 0$, (b) $B = 0.1$, and (c) $B = 0.2$. 
Figure 15: Edge-state properties for a ballistic wire 20 lattice constants wide for magnetic fields of (a) $B = 0.1$ and (b) $B = 0.2$. Distance (number of lattice constants) of the wavefunctions from the center ($<y>$) and wavefunction extents ($<y^2> - <y^2>^{1/2}$) are given as a function of Fermi wave vector. The spin-degenerate Landau-level index ($n$) is used to denote the corresponding edge state.
level approaches a bulk Landau-level energy, however, $l_{loc}$ of the highest (innermost) Landau level grows to become comparable with the wire width $W$.\footnote{This, of course, assumes $W \ll L_{in}$.} There is now significant backscatter and the highest Fermi-level edge states become longitudinally localized\footnote{Once again, if the sample length $L$ is less than $L_{in}$, then the condition for being localized longitudinally becomes $L_{loc} \ll L$.} ($L_{loc} \ll L_{in}$). Depending on the amount of interchannel scattering, lower Landau levels can also be backscattered and localized.

Quantum Hall effect transport, therefore, requires the suppression of intrachannel scattering, that is, scattering between opposite side of the wire. Interchannel scattering, however, can still occur among edge states on the same side of the sample\footnote{}.

There has been considerable interest lately in the nature of this interchannel scattering as a means of equilibrating the current distribution of edge states. In the case of high-field adiabatic transport in high-mobility 2DEG's, this interchannel scattering can also be suppressed. The elastic and inelastic scattering lengths in the longitudinal direction can become very long ($\sim 80 \mu m$ or more). Recent experimental work\cite{23, 25} indicates that this is particularly true of the highest and innermost Landau level (the $N$th level in an $N$-edge-channel system). The $N$th Landau level decouples from the other levels as its center approaches the Fermi energy and the state becomes strongly extended into the bulk. At these particular energies, just as the highest Landau level begins to become backscattered, the scattering amplitude between it and lower-lying (outermost) edge states is reduced. This reduction is expected to become more pronounced with higher fields and with lower disorder\footnote{In the calculations presented in Section 4.4, this hypothesis can be verified directly. In the presence of such a decoupling, a disordered quantum wire should have a monotonically increasing step-like two-terminal conductance as a function of Fermi energy. If the highest Landau level were instead coupled to the lower Landau levels, backscatter of lower Landau levels in addition to the $N$th Landau level would occur as the $N$th Landau level were becoming occupied with increasing Fermi energy. Minima would precede each increase to the next quantized value of the conductance.}. In the calculations presented in Section 4.4, this hypothesis can be verified directly. In the presence of such a decoupling, a disordered quantum wire should have a monotonically increasing step-like two-terminal conductance as a function of Fermi energy. If the highest Landau level were instead coupled to the lower Landau levels, backscatter of lower Landau levels in addition to the $N$th Landau level would occur as the $N$th Landau level were becoming occupied with increasing Fermi energy. Minima would precede each increase to the next quantized value of the conductance.
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4.4 Results of the calculation

Figure 16 shows the two-terminal conductance of a disordered wire plotted as a function of Fermi wave vector in units of $\pi/W$. Curves are offset for clarity. The wire is 20 lattice sites wide ($W = 20$) and the disordered region is 80 lattice sites long ($L = 80$). Results for disorder potentials of $V = 0$, 0.05, 0.1, and 0.2 and magnetic fields of 0, 0.1, and 0.2 are calculated.

The curves at $B = 0$ show the destruction of the quantized ballistic resistance with increasing disorder, consistent with earlier results[93]. In the case of very high disorder, the conductance finally degenerates into fluctuations of order $e^2/h$, the universal conductance fluctuations discussed in Section 1.2.1.

The application of a magnetic field is clearly seen to suppress backscatter. The essential features of this process can be understood by examining the curves corresponding to the disorder potential $V = 0.1$. At $B = 0.1$, conductance quantization has been restored for the first three Landau levels. This is what one would predict from the overlap of the ballistic edge-state wave functions (see Figure 15). Higher Landau levels have edge states that are not confined as closely to the edges of the system and significantly overlap across the width of the wire. Of course, at $B = 0.2$, there is once again quantization of the first three Landau levels. When the disorder potential is increased to $V = 0.2$, quantization is now completely lost at $B = 0.1$. Disorder has hybridized the edge states so that they extend across the width of the wire. At $B = 0.2$, there are two quantized conductance plateaus, but the transition between plateaus shows significant disorder-related fluctuations. When the disorder potential is decreased to $V = 0.05$, there are three well-defined plateaus at $B = 0.1$ and at least four at $B = 0.2$.

For $V = 0.1$ and $B = 0.1$, one notices that it requires a higher Fermi energy to reach the quantized plateaus in the presence of disorder because of the strong backscatter of states near the center of the wire. This is conceptually equivalent to the levitation of the Landau levels in the bulk two-dimensional electron gas (cf. Equation 96 for $\omega_c \tau \sim 3$). When the field is increased to $B = 0.2$, for the same disorder potential ($V = 0.1$), there is some disorder-related structure in the rise to
Figure 16: Two-terminal conductance of a disordered quantum wire with $W = 20$ and $L = 80$ as a function of Fermi wave vector for disorder parameters of $V = 0.0$, $0.05$, $0.1$, and $0.2$ and magnetic field of (a) $B = 0.0$, (b) $B = 0.1$, and (c) $B = 0.2$. Successive curves are offset two conductance units for clarity.
the first Landau level, but no floating is observed ($\omega_c \tau \gg 1$). When the disorder is increased to $V = 0.2$ at $B = 0.2$, however, significant levitation is observed.

The decoupling of the $N$th Landau level from the other $(N-1)$ Landau levels is found to be more pronounced at high fields and less effective in the presence of strong disorder. At $V = 0.1$ and $B = 0.1$, the minima between quantized conductance plateaus indicate backscatter of lower Landau levels and incomplete decoupling of the highest occupied Landau level. When the field is increased to $B = 0.2$, the minimum between the first and second Landau levels disappears, indicating effective decoupling. In the weak disorder case ($V = 0.05$), however, there is substantial decoupling of the highest Landau level even at $B = 0.1$ and no conductance minima are observed.

4.5 Summary

This chapter has shown the suppression of backscatter as the mechanism for the transition to the integral quantum Hall effect regime in quasi-one-dimensional systems. The decoupling the the $N$th Landau level from the other $(N-1)$ Landau levels is shown to be more pronounced at higher fields and in the presence of lower disorder. Levitation of the quantized plateaus with increasing disorder is observed.
Chapter 5

Spin Splitting in Quantum Wires

5.1 Introduction

This chapter uses the Landauer-Büttiker formalism and the edge-state picture discussed in Chapter 4 to consider spin splitting in magnetic fields perpendicular to the plane of the two-dimensional electron gas (2DEG) in the presence of quasi-one-dimensional confinement[94].

The exchange enhancement of the Landé $g$ factor is an interesting consequence of many-body interactions in 2DEG's. First observed in experiments in silicon metal-oxide-semiconductor (MOS) devices[95], the effect was explored theoretically by Janak [96] and by Ando and Uemura[97]. If the numbers of electrons with up and down spins differ in a Landau level, there is a corresponding difference in the exchange energy for each spin orientation which enhances the splitting. Quantitatively, this exchange energy difference is of the form[98]: $E_{ex} = E_{ex}^o(n_{N\sigma=1} - n_{N\sigma=-1})$ where $E_{ex}^o$ scales as the electron-electron interaction energy $e^2/4\pi\epsilon\epsilon_0lb^1$ with a characteristic length scale for the interactions given by the magnetic length $lb = (\hbar/eB)^{1/2}$. Here, $n_{N\sigma=1}$ and $n_{N\sigma=-1}$ denote the relative populations of up and down spins in the $N$th Landau level for a given Fermi-level position. The $g$ factor is, therefore, an oscillatory function of the Fermi energy through the difference $(n_{N\sigma=1} - n_{N\sigma=-1})$ as shown in

\footnote{mks units are used in Chapters 5 and 6. Elsewhere, cgs units are used, which are (unfortunately) more common in the literature.}
Figure 17. The maxima occur when the Fermi level lies between spin-split states.

Figure 17: Exchange-enhanced spin splitting in the two-dimensional electron gas. Shown are energy levels with and without Zeeman splitting for two different Fermi energies. (a) The maximum exchange-enhanced spin splitting occurs when the Fermi energy lies between two spin-split levels. \( g^{*}_{\text{max}} \) is the maximum exchange-enhanced \( g \) factor. (b) The minimum exchange-enhanced spin splitting occurs when the Fermi energy lies between two Landau levels. \( g_{o} \) is the bare \( g \) factor.

For a quantum wire with spin-split quantum channels, the two-terminal conductance \( g \) is given by:

\[
g = \frac{e^2}{h} \sum_{mn} \int dE \left(-\frac{\partial f}{\partial E}\right) T_{mn}(E, B) \tag{97}
\]

The \( T_{mn} \) are transmission probabilities at energy \( E \) and magnetic field \( B \) to go from state \( m \) (a given transverse state of given spin polarization) in one lead to state \( n \) in the other lead. Most of the experimental observations of spin splitting in quasi-one-dimensional systems have been made in ballistic wires. In this case, the two-terminal conductance in the absence of a magnetic field is quantized in units of \( 2e^2/h \)[18, 19]. At high fields, however, the two-terminal magnetoconductance of ballistic constrictions shows spin splitting with conductance plateaus emerging at odd multiples of
$e^2/h$[80, 16]. These systems have the property that, because of the electrostatic confinement, the conductance is quantized even in the absence of a magnetic field perpendicular to the plane of the wire. Recent experiments have used this fact to examine the spin splitting of quantum wires in parallel fields[99]. The effects described necessarily require the existence of a quantizing perpendicular magnetic field and cannot be observed in parallel fields. They are, therefore, the natural extension of experiments in wide 2DEG systems. Recent transport spectroscopy measurements of the Coulomb dot have also raised questions about the enhancement of the $g$ factor in confined geometries[100].

Kinaret and Lee[101] have recently provided theoretical arguments describing how exchange-enhanced spin splitting might become weakened as the width is reduced in a ballistic quasi-one-dimensional wire in a perpendicular magnetic field. They calculate the unscreened exchange interaction, minimizing the total energy for a fixed (1D) electron density. Above a certain critical density, the cost in kinetic energy associated with adding electrons to only one spin-split level increases to a point where it becomes favorable to achieve the same linear density by populating both spin-split levels. At this critical density, a phase transition occurs and the exchange enhancement of the spin splitting collapses. The features of this behavior are shown in Figure 18.

This chapter considers ballistic quasi-one-dimensional wires that are sufficiently wide and magnetic fields that are sufficiently strong that one begins to populate the upper spin split level with increasing linear density, before the onset of this phase transition (i.e. $W > W_{\text{crit}}$ in Figure 18). In this regime, the spin splitting of the Landau levels is enhanced by the presence of confining walls. It is suggested that this is due to an effect in which electron-electron interactions are strengthened because of the interactions with polarized edge-state charges. The experimental system under consideration is one in which electrons are confined to wires in the 2DEG of an $AlGaAs/GaAs$ heterostructure. This quasi-one-dimensional confinement can be achieved with electrostatic gating, etching[102], or low-energy ion exposure[34]. Hard-wall boundary conditions are used, which are probably most accurate for wider channels[103].
Figure 18: Unscreened exchange-enhanced spin splitting in the quantum wire. $E - k$ relations for a single spin-split Landau level for a fixed magnetic field and Fermi energy. As the width of the wire decreases, the linear density in the wire increases and the electrons acquire more kinetic energy. Above a certain critical (1D) density, which occurs at a critical width $W_{\text{crit}}$, the cost in kinetic energy associated with adding electrons to only one spin-split level increases to a point where it becomes favorable to achieve the same linear density by populating both spin-split levels. A phase transition occurs and the exchange enhancement of the spin splitting collapses.
5.2 Calculation of single-particle energy spectrum

5.2.1 Concept of antiscreeing

Any discussion of exchange interactions in quantum wires must consider "screening". In a homogeneous system, an electron polarizes Fermi-level charge in its vicinity, creating a "hole" which in turn acts to screen the electron's Coulomb potential. There is a fundamentally unique nature to the "screening" for quantum wires in the presence of perpendicular magnetic fields. In this case, because of the formation of edge states, polarizable charge is localized within a magnetic length of the wire edge for a Fermi level positioned between Landau levels. Consider first the effects of this "screening" charge on the exchange energy at the subband minimum \((k = 0)\). In this case (see Figure 19a), one is considering the interactions between an electron with wavefunction centered in the middle of the wire \((y = 0)\) and the other electrons in the wire. The \(k = 0\) electron repels electrons in the edges and polarizes them to yield "holes". These "holes", in addition to the electron inducing the polarization, interact with the other electrons in the wire. These electron-electron interactions are enhanced or antiscreeed by the action of the polarization, rather than reduced as in the case of screening in the homogeneous system. The polarized charge is non-local and cannot act, as in the homogeneous case, to screen the Coulomb interaction in the vicinity of the test charge.

This mechanism gives the self-energy dispersion. The self-energy for an electron with positive \(k\) (see Figure 19b) corresponds to interaction with an electron centered in the upper half of the wire \((y = y_o)\), which polarizes more charge in the upper than in the lower edge.\(^2\) Electrons with \(y > y_o\) undergo a stronger enhancement of the repulsive electron-electron interaction because of the larger induced charge in the upper edge and the smaller counterbalancing charge in the lower edge. Electrons with \(y < y_o\) undergo a weaker enhancement due to interaction with the weaker induced charge in the lower edge and the counterbalancing influence of the larger upper edge-state "hole". As one moves away from the subband minima, more electrons undergo the weaker enhancement and fewer undergo the stronger enhancement. The number

\(^2\)The relationship between \(y_o\) and \(k\) is examined in Figure 15.
Figure 19: Physical mechanism for antiscreening (a) for electron with longitudinal momentum $k = 0$ (b) for an electron with longitudinal momentum $k > 0$
effect dominates and the overall enhancement is weaker. This dispersion reduces the
density of states in the center of the wire. This is a natural consequence of the fact
that electrons more strongly repel one another there in the presence of edge-state
polarizability. Also, because the upper edge charge is larger than the lower edge
charge for $k > 0$, the electron centered at $y = y_o$ wants to decay into states closer to
the upper edge. In this case, the self-energy acquires an imaginary part.

5.2.2 “Screened” Hartree-Fock calculation

To confirm these effects, the single-particle energy spectrum, including exchange,
is computed for a ballistic quantum wire of width $W$ in the screened Hartree-Fock
approximation. Screening is calculated in the random-phase approximation. The
Hamiltonian for the system in the presence of a perpendicular magnetic field can be
written as:

$$H = H_o + H_{e-e}.$$  \hspace{1cm} (98)

$$H_o = \sum_{\alpha k} \sum_{\sigma} E_{\alpha k\sigma} a_{\alpha k\sigma}^\dagger a_{\alpha k\sigma}$$ \hspace{1cm} (99)

is the Hamiltonian in the absence of electron-electron interactions. $E_{\alpha k\sigma}$ is the energy
of the magnetoelectric subband modified by the Zeeman splitting,

$$E_{\alpha k\sigma} = E_{\alpha k} - \sigma \mu_B H \frac{g_o}{2}.$$ \hspace{1cm} (100)

$\alpha$ is the magnetoelectric subband index, $k$ is the longitudinal momentum along the
length of the wire in the $x$-direction, $\sigma$ is the spin index which can take on values
of $\pm 1$, and $g_o$ is the “bare” $g$-factor, which has a value of 0.4 near the band edge of
GaAs[98]. $H_{e-e}$ describes the electron-electron interaction. Introducing an electron
field operator,

$$H_{e-e} = \frac{1}{2} \int dx \int dy \int dx' \int dy' \psi^\dagger(x, y) \psi^\dagger(x', y') V(x - x', y - y') \psi(x', y') \psi(x, y)$$ \hspace{1cm} (101)

where

$$V(x - x', y - y') = \frac{e^2}{2\pi \varepsilon_0 \varepsilon \sqrt{(x - x')^2 + (y - y')^2}}.$$ \hspace{1cm} (102)
\( \tilde{\varepsilon} = 12.17 \), the average of the dielectric constants of GaAs and Al\(_{0.34}\)Ga\(_{0.66}\)As, which takes into account the image potential due to the interface with the wide-bandgap semiconductor. The electron-electron interaction is Fourier transformed:

\[
V(x - x', y - y') = \sum_{q_x, q_y} \hat{V}_0(q_x, q_y) e^{iq_x(x-x')} e^{iq_y(y-y')} \tag{103}
\]

\[
= \frac{1}{(2\pi)^2} \int dq_x \int dq_y \hat{V}_0(q_x, q_y) e^{iq_x(x-x')} e^{iq_y(y-y')} \tag{104}
\]

where

\[
\hat{V}_0(q_x, q_y) = \frac{e^2}{2\varepsilon_0 \sqrt{q_x^2 + q_y^2}}. \tag{105}
\]

The eigenstates of the wire in the presence of the magnetic field are given by \( \phi_{\alpha k}(x, y) = \chi_{\alpha k}(y) e^{ikx}, \) where the transverse states are normalized such that \( \int_{-W/2}^{W/2} |\chi_{\alpha k}(y)|^2 \, dy = 1. \) Expanding the field operators in these states gives:

\[
H_{e-e} = \frac{1}{2} \sum_{\alpha k} \sum_{\beta k'} \sum_{\gamma k''} \sum_{\delta k'''} \sum_{\sigma} \sum_{\sigma'} \int_{-\infty}^{\infty} dx \int_{-W/2}^{W/2} dy \int_{-\infty}^{\infty} dx' \int_{-W/2}^{W/2} dy' \times \phi_{\alpha k}(x, y) \phi_{\beta k'}(x', y') \phi_{\gamma k''}(x', y') \hat{V}_0(q_x, q_y) e^{iq_x(x-x')} e^{iq_y(y-y')} \times a^{\dagger}_{\alpha k} a_{\beta k'} a_{\gamma k''} a_{\delta k''} \tag{106}
\]

The corresponding Feynman diagram is shown in Figure 20a. This interaction Hamiltonian can be used to derive the Feynman rules.

The first-order direct contribution to the propagator is assumed to be cancelled completely by a uniform background of positive charge. Previous Hartree calculations show that this is not exactly correct[104]. The requirement that the electron density goes to zero creates a net positive charge at the edges of the system. This charge is balanced by a peak in the electron density approaching the edge, creating local dipoles. Direct energy corrections will modify the energy spectrum slightly but will offer little change to the results presented here.

The first-order exchange contribution to the propagator for the subband \( \alpha \) is shown in the Feynman diagram of Figure 20b. Computing,

\[
\Sigma_{\alpha k}(E) = i \sum_{E'} \sum_{q_x, q_y} \sum_{\beta} \hat{V}(q_x, q_y, E - E') J_{\alpha k, \beta k - q_x}(q_y) J_{\alpha k, \beta k - q_x}(q_y') G_{\beta k - q_x \sigma}(E') \tag{107}
\]
Figure 20: Feynman diagrams (a) for the electron-electron interaction term of the Hamiltonian (b) for the first-order exchange contribution to the propagator, and (c) for the first-order contribution to the polarization
\[ \Sigma_{\alpha \sigma}(E) = \frac{i}{2\pi} \int dE' \sum_{q_x, q_y} \sum_{\beta} \tilde{V}(q_x, q_y, q_y', E - E') J_{\alpha k, \beta k - q_z}(q_y) J_{\alpha k, \beta k - q_z}(q_y') \times \frac{1}{E' - E_{\beta k - q_z}\sigma - \Sigma_{\beta k - q_z}\sigma(E')} \] (108)

where \( \tilde{V}(q_x, q_y, q_y', E - E') \) is the "screened" electron-electron interaction and

\[ J_{\alpha k, \beta k - q_z}(q_y) = \int_{-W/2}^{W/2} dy \chi_{\alpha k}(y)e^{iq_yy} \chi_{\beta k - q_z}(y) \] (109)

In the static approximation,

\[ \Sigma_{\alpha \sigma} = i \sum_{q_x, q_y} \sum_{\beta} \tilde{V}(q_x, q_y, q_y', 0) J_{\alpha k, \beta k - q_z}(q_y) J_{\alpha k, \beta k - q_z}^*(q_y') G_{\beta k - q_z}(t = 0^-) \] (110)

\[ = \sum_{q_x, q_y} \sum_{\beta} \tilde{V}(q_x, q_y, q_y', 0) J_{\alpha k, \beta k - q_z}(q_y) J_{\alpha k, \beta k - q_z}^*(q_y') \times \lim_{\eta \to 0^+} T[a_{\beta k - q_z\sigma}(-\eta)a_{\beta k - q_z\sigma}^+(0)] \] (111)

\[ \Sigma_{\alpha \sigma} = - \sum_{q_x, q_y} \sum_{\beta} \tilde{V}(q_x, q_y, q_y', 0) J_{\alpha k, \beta k - q_z}(q_y) J_{\alpha k, \beta k - q_z}^*(q_y') f_{\beta k - q_z\sigma} \] (112)

\[ = -\frac{1}{(2\pi)^3} \sum_{\beta} \int dq_x \int dq_y \int dq_y' \tilde{V}(q_x, q_y, q_y', 0) \times J_{\alpha k, \beta k - q_z}(q_y) J_{\alpha k, \beta k - q_z}^*(q_y') f_{\beta k - q_z\sigma} \] (113)

where \( f_{\beta k - q_z\sigma} \) is the occupancy function.

The screened electron-electron interaction is given by an integral form of Dyson's equation in the random phase approximation:

\[ \tilde{V}(q_x, q_y, q_y', \omega) = \tilde{V}_0(q_x, q_y)2\pi \delta(q_y - q_y') \]

\[- \tilde{V}_0(q_x, q_y) \sum_{q_y''} \Pi(q_x, q_y, q_y'', \omega)\tilde{V}(q_x, q_y'', q_y', \omega) \] (114)

\[ = \tilde{V}_0(q_x, q_y)2\pi \delta(q_y - q_y') \]

\[- \tilde{V}_0(q_x, q_y)\frac{1}{2\pi} \int dq_y'' \Pi(q_x, q_y, q_y'', \omega)\tilde{V}(q_x, q_y'', q_y', \omega) \] (115)

where \( \Pi(q_x, q_y, q_y', \omega) \) is the polarization, given by

\[ \Pi(q_x, q_y, q_y', \omega) = \sum_{\sigma} \Pi^\sigma(q_x, q_y, q_y', \omega) \] (116)
Computing the first-order contribution to the polarization from the diagram in Figure 20c,

\[
\Pi^\sigma(q_x, q_y, q'_y, \omega) = -i \sum_\alpha \sum_\beta \sum_k \frac{1}{2\pi} \int dE J_{\alpha k, \beta k+q_x} (q_y) J^*_{\alpha k, \beta k+q_x} (q'_y) \times G_{\alpha k\sigma}(E) G_{\beta k+q_x\sigma}(\omega + E)
\]

\[
\Pi^\sigma(q_x, q_y, q'_y, \omega) = \sum_\alpha \sum_\beta \sum_k J_{\alpha k, \beta k+q_x} (q_y) J^*_{\alpha k, \beta k+q_x} (q'_y) \times \frac{f_{\beta k+q_x\sigma} - f_{\alpha k\sigma}}{\omega - (E_{\beta k+q_x\sigma} - E_{\alpha k\sigma}) - (\Sigma_{\beta k+q_x\sigma} - \Sigma_{\alpha k\sigma}) + i0^+}
\]

(117)

(118)

where the occupancy functions are evaluated at energies modified by the self-energy corrections.

In the single-band approximation, only the lowest spin-split Landau level is considered. The further assumption in Equation 118 that \( q_z \) is small yields the Thomas-Fermi-like component of screening. At zero temperature, this results in a considerably simplified expression for the DC polarizability:

\[
\Pi^\sigma(q_x, q_y, q'_y, 0) = \sum_{k = \pm k_F} \frac{1}{2\pi |\partial E_{k\sigma}/\partial k|} J_{k, k+q_x} (q_y) J^*_{k, k+q_x} (q'_y)
\]

(119)

Subband indices have been dropped since one is considering only a single band. In this approximation, it is immediately evident that the polarizing charge is localized at the edges and is proportional to the one-dimensional density of states at the Fermi energy. The polarization is explicitly real. Equations 113, 114, 116, and 119 are solved numerically. \( k_F \) and \( \partial E_{k\sigma}/\partial k|_{k = k_F} \) are determined self-consistently.

### 5.2.3 Results of the calculation

Figure 21 shows the single-particle energy spectra for the lowest Landau level at a magnetic field of 2 \( T \) in the absence of electron-electron interactions and in the presence of unscreened and self-consistently "screened" electron-electron interactions for various wire widths. The real part of the self-energy is explicitly displayed. The Fermi level is positioned at the minimum of the upper spin-split state. The Fermi level defines the zero of energy in Figure 21, and the upper spin-split state is unoccupied.
Figure 21: Calculated single-particle energy spectra of the lowest Landau level for a magnetic field of 2 $T$ and wire widths of (a) 400 nm, (b) 300 nm, (c) 200 nm, and (d) 100 nm. The Fermi level defines the zero of energy. The upper spin-split state is unoccupied. The solid lines show the spin-split Landau level in the absence of electron-electron interactions. The dash-dotted lines show this level in the presence of “screened” interactions.
The qualitative features discussed above are evident in the calculation. In the wider wires, the zero-velocity bulk states have an "unscreened" exchange energy of $-\sqrt{\pi/2e^2/\epsilon l_B}$, the expected two-dimensional value. Polarized edge-state charges produce a strong antiscreening enhancement near the subband minima. The magnitude and extent of this enhancement around $k = 0$ is relatively independent of width up to the widest widths calculated, 600 nm, indicative of the long-range nature of the Coulomb interaction. Figure 20 only shows results up to 400 nm.

The single-particle states are not exact eigenstates of the many-body Hamiltonian for non-zero longitudinal momenta and, hence, decay with characteristic time $\hbar/Im\Sigma$. Figure 22 shows the corresponding imaginary part of the self-energy for a width of 300 nm. All of this decay is due to electron-electron interactions modified by the presence of edge-state charge. This is demonstrated by the fact that in the absence of polarizing charge, Eq. 112 reduces to:

$$\Sigma_{\alpha k\sigma} = -\frac{1}{(2\pi)^2} \sum_{\beta} \int dq_x \int dq_y \tilde{V}_0(q_x, q_y) |J_{\alpha k\beta k-q_x}(q_y)|^2 f_{\beta k-q_x\sigma}$$  \hspace{1cm} (120)

which is explicitly real. For $k > 0$, the upper edge charge is larger than the lower edge charge; electrons are attracted to the upper edge and decay occurs into states of higher $k$. For $k < 0$, the lower edge charge is larger than the upper edge charge and decay occurs into states of lower $k$. The state $k = 0$ is unique. In this case, the upper and lower induced edge-state charges are equal and there is no decay. This follows from Equation 113 with the use of the symmetry $\tilde{V}(q_x, q_y, q_y', 0) = \tilde{V}(q_x, q_y', q_y, 0)$.

### 5.3 Connection with experiment

#### 5.3.1 Measuring g-factor enhancement in quantum wires

In the two-dimensional electron gas, spin-splitting can be observed in the components of the resistivity tensor, $\rho_{xx}$ and $\rho_{xy}$ (or the components of the conductivity tensor, $\sigma_{xx}$ and $\sigma_{xy}$). Minima occur in $\rho_{xx}$ and $d\rho_{xy}/dB$ as a function of $B$ (and similarly, $\sigma_{xx}$ and $d\sigma_{xy}/dB$) when the Fermi level lies between two spin-split states. This is, of course, in addition to the minima that occur when the Fermi energy lies between two
Figure 22: Imaginary part of the self-energy for a magnetic field of $2\, T$ and wire width of $300\, nm$. Values are computed for longitudinal momenta up to the Fermi wave vector.
Landau levels. In the two-dimensional case, $\rho_{xx}$ becomes nonzero when the Fermi-level states become extended, that is, when their localization length $\xi$, introduced in Equation 95, exceeds the inelastic scattering length, $L_{in}$. Therefore, minima (and zeros) in $\rho_{xx}$ (and $d\rho_{xy}/dB$) occur as the Fermi-level states become localized. With increased spin-splitting, minima in $\rho_{xx}$ and $d\rho_{xy}/dB$ become more pronounced. In this case, the overlap of the disorder-broadened spin-split levels at the Fermi energy decreases and $\xi$ consequently decreases.

A similar picture can be carried over to the context of narrow wires. In this case, the relevant transport coefficient is the two-terminal conductance of the wire $g$ (or the corresponding two-terminal resistance $R$). In quasi-one-dimensional systems, it is most useful to use a picture in which only edge states at the Fermi energy carry current. For quasi-one-dimensional systems, it is judicious to use the transverse ($l_{loc}$) and longitudinal ($L_{loc}$) localization lengths for the edge states introduced in Section 4.3. $l_{loc}$ increases as one approaches a subband bottom, depending on the extent of disorder[23]. States are extended in the transverse direction and are backscattered when $l_{loc}$ becomes comparable to the wire width[73]. 3 In general, $dg/dB$ (or $dR/dB$) will, therefore, show oscillatory behavior as a function of $B$ as the Fermi level sweeps through each spin-split subband in the same manner that $\rho_{xx}$ and $d\rho_{xy}/dB$ show oscillatory behavior in the 2DEG as the Fermi energy passes through spin-split Landau levels.

The single-particle energy spectra calculated in Figure 21 show the existence of an antiscreened exchange enhancement around $k = 0$. Referring to Figure 23, one sees that for wide wires approaching the limit of a 2DEG, the enhancement affects only a small fraction of the bulk Landau-level states. A very high density of states persists at the Landau-level energy $E_{LL}$. Significant backscatter occurs for states with energies approaching $E_{LL}$ because of delocalization of edge states due to hybridization with the very high density of bulk Landau-level states. As the wire width is decreased, the antiscreened electrons represent an increasingly larger fraction of the bulk Landau-level states and the high density of states at the energy $E_{LL}$ is removed. Significant

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3This implicitly assumes that the wires are also quasi-one-dimensional in the sense $W \ll L_{in}[35]$. These states, however, become localized in the longitudinal direction ($L_{loc} \ll L_{in}$) as they are backscattered.
Figure 23: The antiscreening enhancement of the single-particle energy spectra for wide and narrow wires. Shown are the $E - k$ relations for a lower spin-split Landau level. The upper spin-split Landau level is unoccupied. The localization length of edge states is denoted schematically. The center two graphs show the components of the conductance for the two cases — the transmission, $T(E)$, weighted by the derivative of the Fermi function, $-\partial f / \partial E$. 
backscatter now occurs for states at a lower energy, closer to the subband minimum. The contribution of this quasi-one-dimensional channel to the conductance is given by 
\[ g = \frac{e^2}{h} \int dE \left( -\frac{\partial f}{\partial E} \right) T(E). \]
Figure 23 shows schematic plots of \( T(E) \) and \( -\partial f/\partial E \) for the narrow and wide wire cases. For a fixed Fermi energy, \( dg/dB \) will be smaller in the narrow case than in the wide case. Spin splitting as determined by the strength of minima in \( dg/dB \) or \( dR/dB \) is, therefore, sensitive to the effects of antiscreening.

5.3.2 Preliminary experimental results

To provide at least some preliminary experimental support for these ideas, quantum wires defined by a split gate of width 500 nm and length 700 nm are fabricated on a conventional \( GaAs/AlGaAs \) two-dimensional electron gas heterojunction (\( n_s = 3.0 \times 10^{11} \) \( cm^{-2} \), \( \mu = 0.98 \times 10^6 \) \( cm^2 / V \) sec). \( \rho_{xx} \) and \( \rho_{xy} \) for a 50\( \mu \)m Hall bar are shown in Figure 30. The details of the device design are discussed in Section 6.2.2. This experiment uses the open cross junction introduced in Figure 31 with only two of the gates biased. In this case, the device and measurement can be represented conceptually as shown in Figure 24. Current flows between contacts 2 and 3. The voltage is measured between contacts 1 and 4. From Equation 62, one sees that this measurement yields the two-terminal resistance of the constriction for magnetic fields directed upward.

Figure 25 shows the two-terminal conductance measured in this manner at 1.8 K as a function of the negative gate bias used to define the wire. The wire is slightly disordered but shows hints of conductance quantization. Figure 26 is a plot of \( dR/dB \) for the five bias points, a – e, noted in Figure 25. A Landau-level index, \( \nu \), is associated with each minimum. The effective two-dimensional density of carriers in the wire is approximately \( 1.9 \times 10^{11} cm^{-2} \) for this range of biases. Table 1 shows the effective wire width assuming hard-wall boundary conditions and weak disorder for each bias point.\(^4\) One notes in Figure 26 the development of a minimum at \( \nu = 5 \) as the width of the wire is decreased from 430 \( nm \) to 230 \( nm \) at a magnetic field of approximately 2 \( T \). This effect is observed on many different constrictions and corresponds to an antiscreening

\(^4\)The exact details of the potentials within gated constrictions are discussed in Section 6.4.3.
Figure 24: Four-terminal measurement of the two-terminal conductance of a split-gate quantum wire in the presence of a magnetic field. The choice of current and voltage contacts shown will give the two-terminal conductance for magnetic fields directed upward.

<table>
<thead>
<tr>
<th>Bias point</th>
<th>Gate voltage (V)</th>
<th>Width (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-0.5</td>
<td>430</td>
</tr>
<tr>
<td>b</td>
<td>-0.7</td>
<td>340</td>
</tr>
<tr>
<td>c</td>
<td>-0.8</td>
<td>320</td>
</tr>
<tr>
<td>d</td>
<td>-0.9</td>
<td>260</td>
</tr>
<tr>
<td>e</td>
<td>-1.0</td>
<td>230</td>
</tr>
</tbody>
</table>

Table 1: Effective wire width for each gate bias used to constrict the quantum wire. The effective 2DEG density is $1.9 \times 10^{11} cm^{-2}$ in the wire.
Figure 25: Two-terminal conductance of a split-gate wire 500 nm wide and 700 nm long as a function of gate bias. Five bias points, a-e, are noted.
Figure 26: Derivative of the two-terminal resistance of the split-gate wire as a function of magnetic field for the five bias points noted in Figure 25.
enhancement of the spin splitting with decreasing wire width. The field and widths correspond closely with the conditions of the calculation. Considerable experimental work remains to be done, including a study of the temperature dependence of the minima and studies in tilted fields. The coincidence method[98] employed to quantify spin splitting in wide 2DEG can be applied to narrow wires by an extension of the analysis of Figure 23.

5.4 Summary

The exchange enhancement of the Zeeman splitting is a well-known consequence of many-body interactions in the two-dimensional electron gas in the presence of a magnetic field. When the electron gas is further squeezed such that the system becomes quasi-one-dimensional, there is a novel enhancement in the repulsive electron-electron interaction, due to interaction with spatially-separated edge-state charges. This antiscreening effect has been confirmed with a “screened” Hartree-Fock calculation of the single-particle energy spectrum. Antiscreening produces increased spin splitting with decreasing wire width as evidenced in oscillations in $dg/dB$ (or $dR/dB$). Preliminary experimental evidence for the effect has also been presented.

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5 The single-band approximation used in this calculation is required to make the problem tractable, while in the experiment there are actually five spin-split channels occupied. This will not qualitatively affect the results presented here. If anything, the additional edge charge will act to strengthen the antiscreening effect calculated.
Chapter 6

Electron Scattering Experiments

6.1 Introduction

The context for understanding transport in ballistic structures has been provided by the Landauer-Büttiker formula discussed in Chapter 2 which treats transport as a scattering problem\cite{38, 39}. The formula's success in describing transport in ballistic structures relies on the fact that electrical conduction is dominated by a single scattering region. The application of the formula to classical ballistic transport is introduced in Section 2.1. The Landauer-Büttiker formula at zero temperature is given by:

\[ \frac{h}{2e} I_i = -N_i \mu_i + \sum_j T_{ij} \mu_j \]  \hspace{1cm} (121)

where \( i \) and \( j \) are indices labelling the leads. \( N_i \) is the number of propagating modes in lead \( i \); \( N_i = \sum_j T_{ij} \). Positive currents imply flow into the leads. The transmission coefficients are given by the transmission probabilities evaluated at the Fermi energy and summed over channel indices:

\[ T_{ij} = \sum_{mn} |t_{ij,mn}(E_F)|^2 \]  \hspace{1cm} (122)

The voltages measured at each reservoir are related to the chemical potentials by the electronic charge, \( V_j = \mu_j / e \). At finite temperature, the transmission coefficients are weighted by the derivative of the Fermi function: \( T_{ij} = \sum_{mn} \int dE \left( -\frac{\partial f}{\partial E} \right) |t_{ij,mn}(E)|^2 \).
In previous experiments, information about the transmission coefficients has been obtained indirectly by resistance measurements, in which fixed currents are applied to the leads and voltages are measured. In a four-probe measurement, a given current is applied at the current leads (into $k$ and out $l$) and the chemical potential difference is measured at the voltage leads (from $m$ to $n$) where the currents are constrained to be zero. As shown in Section 2.3, for the four-probe conductor,

$$ R_{kl,mn} = \frac{h}{2e^2} \frac{T_{mk}T_{nl} - T_{ml}T_{nk}}{D} \quad (123) $$

$D$ is any cofactor of the matrix defined by Equation 121. Because these resistances are complicated functions of the transmission coefficients, they provide only indirect information about the scattering properties of the conductor. A more direct approach is desired.

This chapter presents a new experimental technique for actually measuring the transmission coefficients for ballistic multiprobe microstructures\[105, 106\] and describes samples which have been designed to be almost literal realizations of the Landauer-Büttiker model.

This chapter is organized as follows. Section 6.2 describes the four-probe samples designed specifically for this experiment. Section 6.3 discusses the transmission coefficients obtained for the open cross junction, in which there is no geometrical distinction between the probes. This geometry constitutes a “simplest case” test of the technique. Section 6.4 discusses experiments in a pinched cross junction, in which quantum point contacts are used to separate two of the probes from the main conductor. This facilitates two interested new experiments. First, the geometry allows a momentum spectroscopy of electrons emitted from a quantum point contact. Interesting new quantum effects are present in this distribution which provide details on the nature of conductance quantization in realistic potentials. Second, this configuration is also used to study the perturbation upon the measurement caused by probes in the ballistic regime.
6.2 Samples and experimental details

6.2.1 Measuring the transmission coefficients

As a starting point, one can conceptualize one possible way of measuring the transmission coefficients as shown in Figure 27 for the four-probe open cross geometry. As mentioned above, conventional resistance measurements implicitly involve the imposition of boundary conditions upon the currents, \( I_i \). The current is fixed to flow only between two probes which one designates as current probes. The other probes are used to measure voltages, and, thus, currents through these probes are constrained to be zero. If the chemical potentials are instead fixed, one obtains an immediate simplification in Equation 121. Specifically, one sources a current \( I_1 \) into reservoir 1, increasing its chemical potential to some level \( \mu_1 \), while equalizing \( \mu_i \) at the other

![Figure 27: Conceptual basis for the T-measurement technique. A current \( I_1 \) is sourced into reservoir 1, increasing its chemical potential to some level \( \mu_1 \) while equalizing \( \mu_i \) at the other reservoirs by an ideal short-circuit connection through current preamplifiers.](image-url)
reservoirs by an ideal short-circuit connection. One is thus free to rescale all of the equalized $\mu_i$ in the collection leads such that $\mu_2 = \mu_3 = \mu_4 \equiv 0$. Under these conditions, the currents through these short-circuit terminations are measured with ideal ammeters (cf. Figure 27). Equation 121 simplifies to the form:

$$\frac{h}{2e} \begin{pmatrix} I_1 \\ -I_2 \\ -I_3 \\ -I_4 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{21} & T_{31} & T_{41} \\ T_{12} & T_{22} & T_{32} & T_{42} \\ T_{13} & T_{23} & T_{33} & T_{43} \\ T_{14} & T_{24} & T_{34} & T_{44} \end{pmatrix} \begin{pmatrix} \mu_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - N_1 \begin{pmatrix} \mu_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{124}$$

One immediately obtains the first column of the transmission matrix. The off-diagonal $T_{ij}$ are directly proportional to the currents measured by the ammeters:

$$T_{ii} = \frac{h}{2e \mu_i} I_i \quad (i \neq 1) \tag{125}$$

while

$$(T_{11} - N_1) = \frac{h}{2e \mu_i} I_i \tag{126}$$

The other columns of the transmission matrix can be obtained by changing the source lead and following the same procedure.

In an experiment, the real current paths used to short-circuit the reservoirs contain finite resistances such as those in the wire bonds, contacts, external leads, and current amplifiers. For each path from the reservoirs to an external point defined as "ground", both the resistance and the current carried by each separate path will be unique. The resulting path-specific potential drops render external short-circuit connections ineffective at equalizing the $\mu_i$ which are internal to the sample.

This difficulty is circumvented by including a separate current contact (I-contact) and voltage contact (V-contact) at each reservoir, which enables one to simultaneously monitor all the $\mu_i$ in the presence of current flow. One can now actively null differences between the three chemical potentials of the collection reservoirs and achieve the simplification of Equation 124 dynamically by introducing external, variable terminating impedances between each (sink) I-contact and "ground." This procedure is outlined in Figure 28 for a four-probe device, including a flow chart.
Figure 28: One possible implementation of the T-measurement technique. Arrows show the direction of current flow for B directed upward. Assignment of current (light shading) and voltage (dark shading) contacts depends upon field orientation. For clarity, only connection to reservoirs 1 and 3 are shown.
CHAPTER 6. ELECTRON SCATTERING EXPERIMENTS

For four-probe conductors, this method requires a self-consistent adjustment at each of three reservoirs for every value of magnetic field. In practice, this technique is more difficult than an alternative, mathematically analogous, procedure. This second approach employs static terminations and involves building a set of linear equations, through successive measurements (sweeps), which are ultimately solved to obtain the transmission matrix. Current (typically 10 nA) is injected into the current contacts of the injection reservoir and extracted and measured through current preamplifiers through the current contacts of the three collection reservoirs. The chemical potentials (voltages) of these three collection reservoirs are measured with respect to the injecting reservoir (referenced to \( V = 0 \)). These experiments are performed at 2K using conventional lock-in techniques at a frequency of \( f = 14Hz \). The transmission coefficients are obtained by measuring all the voltages and currents as a function of magnetic field for seven different static current terminations (for the four-probe open cross structure). These terminations are represented in Figure 29 for the case of current introduced via lead 1. They are the complete set of separate configurations by which current can be collected from one (3 configs.), two (3 configs.), or all three (1 config.) remaining leads.

For each magnetic field point, one then proceeds to solve an overdetermined linear system to extract the transmission coefficients. Consider, for example, lead 2 for each of seven configurations (denoted by the index \( \alpha \)). The equation for \( I_2 \) has the form:

\[
I_2^{(\alpha)} = \frac{2e}{h} (T_{22}^{(\alpha)} V_2^{(\alpha)} + T_{23}^{(\alpha)} V_3^{(\alpha)} + T_{24}^{(\alpha)} V_4^{(\alpha)})
\]

where \( T_{ij}' = T_{ij} - N_i \delta_{ij} \). In matrix form, this becomes:

\[
I_2 = \frac{2e^2}{h} VT^2
\]

\[
[7 \times 1] = [7 \times 3][3 \times 1]
\]

The least-squares solution of this linear system is given by:

\[
T_2' = (V^T V)^{-1} V^T I_2
\]

This procedure can then be repeated using data taken simultaneously for leads 1, 3, and 4. One can obtain the remaining column of \( T \), i. e. the \( T_{ii} \)'s, by injecting current
Figure 29: Representation of the seven different static current terminations used to solve for $T$ for the case of current introduced via lead 1. For the case of geometries in which there is only very small transmission through a given lead, the open circuit connection is replaced by a $\approx 10k\Omega$ shunt connection to ground.
into a different lead and repeating this procedure. This approach, in general, provides a set of transmission coefficients satisfying the reciprocity relation of Equation 28 to within a few percent.

Because the system is overdetermined, variances for each transmission coefficient can also be extracted. Specifically, consider the linear system solved to find $T_{42}$:

$$I_4^{(i)} = T_{42}V_2^{(i)} + T_{43}V_3^{(i)} + (T_{44} - N_4)V_4^{(i)}$$  \hspace{1cm} (130)

Data is taken for seven configurations (denoted by the configuration index $i$) to find the three coefficients, $T_{42}$, $T_{43}$, and $T_{44} - N_4$. This leaves four degrees of freedom. All the error in the least-squares fit can be assigned to an "effective" error in measuring the current $I_4$[107]

$$\sigma^2_{I_4} \approx \frac{1}{4} \sum_{i=1}^{7} (I_4^{(i)} - \bar{T}_{42}V_2^{(i)})^2$$  \hspace{1cm} (131)

where $\bar{T}_{42}$ denotes the transmission coefficient determined by the least-squares fit. Similarly,

$$\sigma^2_{\bar{T}_{42}} \approx \frac{1}{4} \sum_{i=1}^{7} \frac{(I_4^{(i)} - \bar{T}_{42}V_2^{(i)})^2}{V_2^{(i)}^2}$$  \hspace{1cm} (132)

This variance data sheds additional new light on the underlying physics of conduction in these structures (see Section 6.4.4).

The choice of I-contact and V-contact must be deliberate. As in the two-reservoir, four-contact device considered in Section 5.3.2, a different choice of contacts must be made for each magnetic field orientation. These conditions constitute a multiprobe generalization of the work of van Houten et al.[67], in which four-terminal resistance measurements are used to measure two-terminal resistances in the presence of a magnetic field (see Equation 62). Arrows show the direction of current flow for a magnetic field directed upward. In the presence of this field, the voltage probe must be in equilibrium with the electrons incoming from the current probe. This requires that current entering the current probe must enter the voltage probe first before it enters the scattering region. This choice of contacts is shown in Figure 28 for magnetic field orientated upward. If the opposite choice is made, a spurious Hall potential drop will result (cf. Equations 61 and 63).
The use of two contacts for each reservoir still leaves a small background resistance of $< 140\Omega$ associated with voltage drops within the wide 2DEG rectangular regions and the "spreading" resistance associated with the coupling of these regions to the narrow gated leads of the junction. In the few-mode regime when $R_{\text{lead}} \geq 3k\Omega$, they result in only few-percent systematic corrections to the measured transmission coefficients.

For the case of geometries in which there is only very small transmission through a given lead, the open circuit connection is replaced by a $\sim 10k\Omega$ shunt connection to ground. This is sufficient to differentiate it from the short circuit, $0\Omega$, configuration. This allows measurement of the transmission matrix even in the presence of nearly vanishing electron flux and constrains the reservoir to avoid spurious drifting in $\mu$. In four-terminal resistance measurements, it is difficult to use a lead that is only weakly-coupled to the scattering region as a voltage probe. The chemical potential of such a "floating" reservoir is determined by ill-defined parasitic leakage paths. In this measurement, however, each reservoir is tied to ground through an impedance in the current shunt leads that is much less than any such parasitic leakage impedance. As described in Section 6.4, this allows "measurement" of four-terminal resistances with weakly-coupled voltage probes.

### 6.2.2 Sample fabrication

The devices for transmission coefficient measurements introduced in Figure 28 are patterned from a conventional modulation-doped GaAs/$Al_{0.3}Ga_{0.7}As$ heterojunction. An undoped $1\mu m$ GaAs buffer is grown by molecular-beam epitaxy on a Cr-doped semi-insulating GaAs substrate. This is followed by an undoped $150\AA$ $Al_{0.3}Ga_{0.7}As$ spacer layer and a $600\AA$-thick $Al_{0.3}Ga_{0.7}As$ layer, doped $1.5 \times 10^{18} cm^{-3}$. Growth is terminated with a $50\AA$ cap layer doped $2 \times 10^{18} cm^{-3}$. The 2DEG density obtained from the Shubnikov-de Haas oscillations in the magnetoresistance is $n_s = 3 \times 10^{11} cm^{-2}$. The impurity-limited mobility is $1.0 \times 10^6 cm^2/V sec$ which corresponds to a transport mean-free path of $l = \frac{\hbar k_F \mu}{e} \sim 9\mu m$. Bulk magnetotransport properties are shown in Figure 30. The Fermi wavelength is $\lambda_F = \left(\frac{2\pi}{n_s}\right)^{1/2} = 46 nm$. 
Figure 30: Longitudinal and Hall resistivities of the 2DEG as measured on 50μm Hall bars at $T = 4.2K$. (a) $\rho_{xy}$ versus magnetic field (b) $\rho_{xx}$ versus magnetic field. Minima associated with fill factors $\nu = 4, 6,$ and $8$ are noted. $n_s = 3 \times 10^{11} cm^{-2}$. The zero-field resistivity gives an impurity-limited mobility of $1.0 \times 10^6 cm^2/V sec$. 
CHAPTER 6. ELECTRON SCATTERING EXPERIMENTS

The device geometry used in these studies is shown in the SEM micrographs of Figure 31. Twelve Au/Ge/Ni/Au Ohmic contacts are defined by optical lithography and lift-off. They are subsequently annealed into the sample to contact the 2DEG. In another mask step, Cr/Au fingers are defined which lead from these contacts. A final optical lithography step defines a central rectangular mesa $20 \times 40 \mu m^2$, which is ion-etched into the semiconductor. Two electron beam lithography steps on the JEOL JBX-5DII follow. Using a single level PMMA mask and exposure to 500 eV Ne$^+$ ions[34], four of the fingers are isolated from the 2DEG and the mesa is divided into four rectangular regions connected at the center. In a second step using a bilevel PMMA process, four Cr/Au gates, 100 nm thick, are added. These define the "scatterer" in the center of the mesa. By applying a negative bias on the gates with respect to the 2DEG, one can deplete the electron gas from beneath the gates and define the open cross junction for the pattern shown in Figure 31.

The structures fabricated are almost literal realizations of the ideal Landauer-Büttiker model. The ion exposure separates the mesa into four rectangular regions which function as reservoirs. Each is designed with two Ohmic contacts, for purposes of the transmission coefficient experiment. Each reservoir feeds a lead, which is formed near the center of the mesa by each adjacent pair of gates. These leads, as defined lithographically by the gates, are 400 nm wide, while the actual channel produced in the 2DEG is less than 300 nm wide. The scattering region, which subsumes all the transport physics of the conductor, is the open cross junction defined by the gates.

6.3 Transport in the open cross junction

The first structure studied is the open cross junction shown in Figure 31, a "simplest case" test of the technique. In particular, there are several low-field ballistic magnetoresistance effects which have received considerable attention. One such effect is the suppression, or quenching, of the Hall resistance around zero field, first observed by Roukes, et al.[13]. When the current is forced between opposite probes of a cross junction, a Hall resistance $R_H$ arises which is proportional to the transverse voltage induced between two opposite probes (see inset of Figure 35). Another effect is the
Figure 31: SEM micrographs of one device used in this study. *Top inset:* Wire bonds attached to twelve Au/Ge Ohmic contacts. Cr/Au fingers lead from these contacts to a central mesa. The devices are \( \sim 1\mu m \) across. *Top right:* Central mesa is \( 20 \times 40\mu m^2 \). Four fingers are isolated from the 2DEG of the mesa by ion exposure. In addition, this ion exposure step divides the mesa into four rectangular regions connected at the center. Four Cr/Au gates are then defined. *Bottom:* These gates form a central cross pattern. Adjacent gates are separated by 400 \( nm \).
negative bend resistance at $B = 0$, which decays with increasing field as first observed by Takagaki et al.\cite{15} and which is related to anomalies seen earlier by Timp, et al.\cite{14} The bend resistance is proportional to the voltage drop between two adjacent probes when current is forced through the two adjacent probes on the opposite side (see inset of Figure 36).

The key components to the physics of transport in junctions have been identified, and they are essentially manifestations of classical ballistic transport\cite{108}, in which the electrons injected by the lead scatter, like billiard balls, from the electrostatic sidewalls of the junction. Experiments show that this scattering can be highly specular\cite{85}. Experiments\cite{108} and calculations\cite{79} suggest that short trajectories dominate the behavior of the transmission coefficients. The open cross junction has been the natural starting point for discussions of mesoscopic transport phenomena because of its $C_{4v}$ symmetry.\footnote{This is the Schoenflies notation, used in solid-state theory to describe crystallographic point groups.} This implies four-fold rotational symmetry and the existence of four mirror planes containing the axis of rotation. It constitutes the simplest four-probe conductor.

In this case, one is able to write simplified expressions for the Hall and bend resistances from Equation 123\cite{79}:

$$R_H = \frac{h}{2e^2} \frac{T_R^2 - T_L^2}{D}$$

$$R_B = \frac{h}{2e^2} \frac{T_LT_R - T_F^2}{D}$$

where $D = (T_R + T_L)[2T_F(T_F + T_R + T_L) + T_R^2 + T_L^2]$. $T_F(= T_{13} = T_{24})$ is the forward transmission coefficient, $T_R(= T_{21} = T_{43})$ are the right-turning coefficients, and $T_L(= T_{41} = T_{23})$ are the left-turning coefficients. The magnetic field dependencies in the transmission coefficients are manifest in the anomalous low-field magnetoresistance characteristics discussed above. The negative bend resistance is caused by the enhancement of the forward transmission over the right- and left-turning transmission coefficients around $B = 0$. The quenching of the Hall effect may be caused by either decreasing the turning probabilities with respect to the forward transmission.
or by decreasing the asymmetry between left and right turning probabilities. Various mechanisms have been proposed by which these may occur.

This technique allows direct measurement of the transmission coefficients for the first time. One is then able to determine the extent to which symmetries hold. The use of four gates to define the junction allows one to compensate for potential variations and bias the device symmetrically. Determining the scattering properties of the junction experimentally allows one to verify that they produce measured anomalous low-field magnetoresistance characteristics through Equation 123.

### 6.3.1 Transmission coefficients

The transmission coefficients for the open cross junction are measured at two different bias points: for bias $a$, $V_g = -0.9V$; and for bias $b$, $V_g = -0.7V$. These biases correspond to values of the Fermi energy resulting in approximately five and seven occupied modes in each of the leads, respectively. Figure 32 presents the results for $T_F = T_{31}$, the transmission coefficient for forward propagation. The curve is peaked at $B = 0$ and falls off monotonically with increasing field as the electron distribution emitted from lead 1 is steered away from collection in lead 3.

Figures 33 and 34 present the right-turning coefficients $(T_R)$, $T_{41}$ and $T_{23}$, and the left-turning coefficients $(T_L)$, $T_{21}$ and $T_{43}$. For magnetic fields oriented such that the Lorentz force steers the electron beam into a given lead, the turning coefficient rises to a level close to the total number of injected modes. It subsequently decreases slightly with increasing field, which can be attributed to magnetic depopulation. For field values such that one would expect the electron beam to be steered away from collection in a given lead, a peak in transmission occurs, associated with specular reflection from the rounded corner of the junction. This is the first direct observation of such a "rebound" trajectory[109], shown in the insets of Figures 33 and 34. Both $T_{21}$ and $T_{43}$ and $T_{41}$ and $T_{23}$ would be strictly equal in the case of $C_{4v}$ symmetry. Differences arise due to the actual asymmetries in the electrostatic (cf. lithographic) junction potential. Comparing the magnitude of $T_{31}$ with the turning probabilities $T_{21}$ and $T_{41}$ near $B = 0$ confirms that forward propagation is favored. This is probably due in part to the collimation of the beam injected by lead 1, discussed in a more
Figure 32: Forward transmission probability $T_{31}$ for the open cross junction. Two bias points are noted: for bias $a$, $V_g = -0.9V$; and for bias $b$, $V_g = -0.7V$. Inset: Ballistic trajectory at zero field.
Figure 33: Right-turning transmission probabilities for the open cross junction at biases $a$ and $b$. Arrows show field regions at which rebound trajectories occur. (a) $T_{21}$ versus magnetic field. Top inset: Ballistic trajectory showing “rebound” into lead 2. Bottom inset: Ballistic trajectory for direct steering into lead 2. (b) $T_{43}$ versus magnetic field. Top inset: Ballistic trajectory showing “rebound” into lead 4. Bottom inset: Ballistic trajectory for direct steering into lead 4.
Figure 34: Left-turning transmission probabilities for the open cross junction at biases $a$ and $b$. Arrows show field regions at which rebound trajectories occur. (a) $T_{41}$ versus magnetic field. *Top inset:* Ballistic trajectory showing "rebound" into lead 4. *Bottom inset:* Ballistic trajectory for direct steering into lead 2.
definitive context in Section 6.4.1.

6.3.2 Hall and bend resistances

The measured transmission coefficients are used to calculate, or "reconstitute", the Hall and bend resistance through Equation 123. These results are compared with the directly measured values in Figures 35 and 36. Small discrepancies between the

![Graph showing the relationship between magnetic field and magnetic field resistance](image)

Figure 35: Hall resistance $R_H = R_{13,24}(B)$ for the open cross junction at biases $a$ and $b$ reconstituted from the measured $T_{ij}$ (solid curves). Also shown is $R_H$, measured conventionally at each bias (dashed curves). The curve for $a$ is offset -0.1. Inset: Choice of current and voltage probes for this measurement.

"reconstituted" and measured resistances arise because the gate bias defining the junction potential must be brought to zero and reapplied between the transmission coefficient experiment and the four-terminal resistance measurement. Changes to
Figure 36: Bend resistance $R_B = R_{12,43}(B)$ for the open cross junction at biases $a$ and $b$ reconstituted from the measured $T_{ij}$ (solid curves). Also shown is $R_B$, measured conventionally at each bias (dashed curves). Inset: Choice of current and voltage probes for this measurement.
the transmission coefficients always result from this procedure, perhaps reflecting the difficulty in reachieving exactly the same confinement potential.

6.4 Transport in the pinched cross junction

This section discusses experiments in a pinched cross junction, in which quantum point contacts are used to separate two of the probes from the main conductor. This facilitates two interesting new experiments. First, the geometry allows a momentum spectroscopy of electrons emitted from a quantum point contact. Second, this configuration is used to study the perturbations upon the measurement caused by probes in the ballistic regime.

Recent work views the quantum point contact as a slowly-varying saddle point potential[110]. Conductance is quantized because the potential in the constriction is locally adiabatic [111], implying that the mode number for the transverse motion is conserved. Each mode that is successfully transmitted contributes one unit of $2e^2/h$ to the conductance. Away from the constriction, the potential necessarily becomes non-adiabatic as it widens to join the 2DEG. Despite intermode scattering there, the probability of backscatter is small and the conductance remains quantized. Section 6.4.3 presents significant modifications to the adiabatic saddle-point picture of the quantum point contact.

The pinched-cross-junction “scatterer” is shown in the SEM micrographs of Figure 37[105]. Leads of a cross junction are, as in the open cross case, defined by gates separated by 400 nm. However, in this case, two of these leads are pinched by 150 nm side constrictions formed between two 50 nm-wide gate fingers. By contrast with the open cross junction, the ideal pinched cross junction has only $C_{2v}$ symmetry, that is two-fold rotational symmetry with two mirror planes containing the axis of rotation. In this case, the Hall and bend resistances are given by:

$$R_H = \frac{h}{2e^2} \frac{T_R \tilde{T}_R - T_L \tilde{T}_L}{D}$$

$$R_B^{(1)} = \frac{h}{2e^2} \frac{T_R \tilde{T}_L - T_F \tilde{T}_F}{D}$$

(135)

(136)
Figure 37: SEM micrographs of the pinched cross junction. Leads of the pinched cross junction are, as in the case of the open cross junction, defined by gates separated by 400 nm. Here, however, two of these leads are pinched by 150 nm side constrictions formed between two 50 nm gate fingers.
\[ R_B^{(2)} = \frac{h}{2e^2} \frac{\hat{T}_RT_L - T_F\hat{T}_F}{D} \] (137)

There are two distinct bend resistance in this case. \( T_{R(L)} \) is the turning coefficient for transport from the main channel right (left) into one of the point contacts. \( \hat{T}_{R(L)} \) is the turning coefficient for transport from the quantum point contact right (left) into the main channel. \( T_F \) is the transmission coefficient from point contact to point contact. \( \hat{T}_F \) is the transmission coefficient down the main channel.

Figure 38 displays the two-terminal conductance across the constrictions as a function of gate bias. The conductance is approximately quantized in units of \( 2e^2/h \).

![Figure 38: Two-terminal conductance across the constrictions as a function of gate bias. Three bias points are labelled a, b, and c.](image)

Three bias points, labelled a, b, and c, are denoted. These biases permit only small transmission through the constriction \( t \sim 0.5, 1.5, \) and 2.5, respectively, while allowing much larger transmission along the main conductor, \( T \sim 7, 8, \) and 10. The properties of the effective potential in the scattering region at these biases is summarized in Table 2. \( n_s \) is the effective two-dimensional electron gas density in the main
<table>
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<tr>
<th>Bias point</th>
<th>$V_g$ (V)</th>
<th>$n_s$ ($cm^{-2}$)</th>
<th>L (nm)</th>
<th>$X^+$ (nm)</th>
<th>$X^-$ (nm)</th>
<th>$W_c$ (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-0.9</td>
<td>$1.8 \times 10^{11}$</td>
<td>210</td>
<td>55.0</td>
<td>50.0</td>
<td>28</td>
</tr>
<tr>
<td>b</td>
<td>-0.84</td>
<td>$2.0 \times 10^{11}$</td>
<td>220</td>
<td>64.5</td>
<td>57.5</td>
<td>90</td>
</tr>
<tr>
<td>c</td>
<td>-0.7</td>
<td>$2.2 \times 10^{11}$</td>
<td>270</td>
<td>115.0</td>
<td>83.0</td>
<td>146</td>
</tr>
</tbody>
</table>

Table 2: Parameters for the pinched junction. $V_g$ is the applied negative gate bias. $n_s$ is the effective two-dimensional electron gas density in the main channel. L is the distance between the constrictions, extracted from the conductance of the leads, assuming hard-wall potentials. $X^+$($X^-$) is the distance of the left (right) reflection point from the center of the collector. $W_c$ is the effective width of the collector.

channel, obtained from the quantum Hall effect. L is the distance between the constrictions, extracted from the conductance of the leads assuming hard-wall potentials ($W = \frac{h\tau}{2e^2k_F}$). The other parameters displayed in Table 2 are introduced below.

### 6.4.1 Forward transmission coefficient and analysis

Molenkamp et al.[112] measured the bend resistance with a series arrangement of point contacts to demonstrate collimation of an electron beam emitted by such a contact. Collimation refers to an electron distribution that is not angularly isotropic but is peaked in the forward direction[113]. This is an effect arising from the invariance of $p_xW$ (or in the few mode case, by the conservation of transverse mode index), in the slowly widening orifice. The increasing width lowers the transverse momentum, thereby increasing the forward momentum.

The experiment here allow forward transmission measurements even in the case of nearly vanishing electron flux. This permits exploration of the details of the injected momentum distribution for the first time in the few-mode quantum limit. Figure 39 is a plot of $T_{42}$ as a function of magnetic field. The curves are sharply peaked near $B = 0$ but show shoulder features for fields $|B| \sim 0.3T$ which were not observed in Figure 32 for the case of the open cross junction.

The starting point for this analysis is the assumption that propagation between the two point contacts can be treated classically. The approach is similar to previous work on classical electron focussing[16]. For each point contact, the quantity $dT/d\theta$

\footnote{An isotropic distribution is given by $P(\theta) = \cos(\theta)/2$.}
Figure 39: Forward transmission coefficient \( T_{42} \) for electrons propagating through both constrictions. *Inset:* Ballistic trajectory at zero field.

is the transmission per unit angle of emission.\(^3\) The total transmission through the point contact is then given by \( T = \int_{-\pi/2}^{\pi/2} \frac{dT}{d\theta} \ d\theta \), which defines its conductance \( g = \frac{2e^2}{\hbar} T \) for spin degeneracy. The emitted electron flux per unit angle is given by \( \frac{\hbar k_{F}}{m^*} \frac{dT}{d\theta} \). Reciprocity of the injector and collector requires that \( dT/d\theta \) also define the maximal collection per unit angle.

The differential transmission per unit angle emitted by the injector (collector) at angle \( \theta \) is given by \( dT_{i(c)}(\theta) \)/\( d\theta \). The collector and injector are assumed to have the same width, \( W_c \). The maximum amount of transmission that is collected by a differential element \( dx_c \) of the collector is given by:

\[
\left| \frac{d\theta'}{dx_c} \right| \frac{dT_c(\theta')}{d\theta'} 
\]

where \( \theta' \) is the angle of collection and \( x_c \) is the coordinate along the collector. For

\(^3\) This quantity referred to as simply \( T(\theta) \) in Shepard, et al. [105].
the case of point injection and point collection \((W_c/L \ll 1)\),

\[
T_F = \frac{W_c}{L} \frac{dT(\theta)}{d\theta}
\]

(139)

where

\[
\frac{dT}{d\theta} = \min \left( \frac{dT_1(\theta)}{d\theta}, \frac{dT_2(\theta)}{d\theta} \right)
\]

(140)

and \(\theta\) is determined by \(L/2 = r_c \sin \theta\). \(r_c = (\hbar/eB)\sqrt{n_s/2\pi}\) is the cyclotron radius and \(L\) is the distance between the constrictions.

The “min of” result is exactly that obtained previously for the case of magnetotransport between two point contacts in series[114]. The quantized ballistic conductance of one point contact is \(g_1 = \frac{2e^2}{h} N_1\) while the other is \(g_2 = \frac{2e^2}{h} N_2\). In the case where transport from one point contact to the other is adiabatic (i.e. inter-Landau-level scattering is suppressed) \(g_{\text{series}} = \frac{2e^2}{h} \min(N_1, N_2)\). In this case, transport is “adiabatic” if the electrons are essentially ballistic, meaning that scattering events involve only small angles. At 2 K, acoustic phonon and electron-electron scattering are the dominant phase-breaking mechanisms. Both are small angle scattering events, which preserve ballistic (“adiabatic”) transport.

Beginning with the data from Figure 39, Equations 139 and 140 and the values for \(W_c\) noted in Table 1 are used to obtain the plot of \(dT/d\theta\) shown in Figure 40. The assumption of point injection and collection is less valid for bias \(c\) where \(W_c/L \sim 0.54\) than bias \(a\) where \(W_c/L \sim 0.13\). \(dT/d\theta\) is an “image” of the injected electron beam. The emitted electrons have a collimated distribution. Distinct side-lobe features, however, are present even for less than one propagating mode.

One might expect to observe corresponding features to those observed in \(T_{42}\) in a straightforward four-probe measurement of the bend resistance, \(R_B^{(1)}(B) = R_{12,43}(B)\) or \(R_B^{(2)}(B) = R_{23,14}(B)\). Figure 41 shows \(R_B^{(1)}(B)\) reconstituted from the measured \(T_{ij}\). Also shown is \(R_B^{(1)}\) measured conventionally at bias \(c\). Only in this least-pinched off case can we actually measure \(R_B\) directly using a four-probe configuration because of the complications arising from use of a weakly-coupled probe to measure voltage. For biases \(a\) and \(b\), this precludes direct measurements and one must settle for reconstituted traces. The large negative value at \(B = 0\) appear to reflect collimation.
Figure 40: $dT/d\theta$, extracted from a semiclassical analysis of the data in Figure 39 (see Equation 129). Parameters for this conversion are shown in Table 2.
Figure 41: Bend resistance, $R_B^{(1)} = R_{12,43}(B)$, for the pinched cross junction at biases $a$, $b$, and $c$ reconstituted from the measured $T_{ij}$ (solid curves). Also shown is $R_B$ measured conventionally at bias $c$ (dashed curves). Curves for $a$ and $c$ are offset by 0.05 and -0.05, respectively. *Inset:* Choice of current and voltage probes for this measurement.
of the beam in the forward direction. Except for the central dip in $R_B$, additional corresponding features between $T_F$ and $R_B$ are almost completely absent in the measured resistance for bias $c$. In the reconstituted resistances for biases $a$ and $b$, some of the side-lobe structure in $T_{42}$ appears to be reflected in corresponding structure in $R_B$. However, in these cases, the correlation is not conclusive because $R_B$ is also a function involving other transmission coefficients. (cf. Equations 136 and 137).

6.4.2 Turning coefficients

Figure 42 displays the turning coefficients, $T_R$, for transport from the main channel right into one of the quantum point contacts, $T_{41}$ or $T_{23}$. Figure 43 displays the turning coefficients, $T_L$, for transport from the main channel left into one of the quantum point contacts, $T_{21}$ or $T_{43}$. Both $T_{41}$ and $T_{23}$ and $T_{21}$ and $T_{43}$ would be strictly equal in the case of $C_{2v}$ symmetry. As in the open cross case, differences arise due to the actual asymmetries of the junction potential. For magnetic fields oriented such that the Lorentz force steers the electron beam into a given lead, the transmission coefficient rises to the total number of injected modes. For field values such that one would expect the electron beam to be steered away from collection in a given lead, a peak in transmission occurs. As in the case of the open cross junction (cf. Figures 33 and 34), this is due to a simple trajectory in which the electron beam is reflected specularly from a small segment of the smooth electrostatic boundary near the constriction and directed into the "wrong" lead as shown in the insets. In the case of bias $a$, the "rebound" peak becomes comparable in magnitude to the fully deflected flux.

Measurement of the momentum distribution of electrons emitted by the point contact is also possible using these turning coefficients. The transmission coefficients which describe transmission from the quantum point contacts into the main channel at lead 2, $\hat{T}_R(B) = T_L(-B) = T_{12}(B)$ and $\hat{T}_L(B) = T_R(-B) = T_{32}(B)$, are used for this analysis. In this case, a small segment of the rounded electrostatic contours near the constriction at lead 4 directly opposite from the injector at lead 2 specularly reflects electron flux into the side probe as shown in the diagrams at the top of Figure 44. As the beam is swept past this reflecting spot, the angular profile of one side of
Figure 42: Right-turning transmission probabilities for the pinched cross junction at biases $a$, $b$, and $c$. (a) $T_{21}$ versus magnetic field. Top inset: Ballistic trajectory showing “rebound” into lead 2. Bottom inset: Ballistic trajectory for direct steering into lead 2. (b) $T_{43}$ versus magnetic field. Top inset: Ballistic trajectory showing “rebound” into lead 4. Bottom inset: Ballistic trajectory for direct steering into lead 4.
Figure 43: Left-turning transmission probabilities for the pinched cross junction at biases \( a \), \( b \), and \( c \). (a) \( T_{41} \) versus magnetic field. Top inset: Ballistic trajectory showing "rebound" into lead 4. Bottom inset: Ballistic trajectory for direct steering into lead 4. (b) \( T_{23} \) versus magnetic field. Top inset: Ballistic trajectory showing "rebound" into lead 2. Bottom inset: Ballistic trajectory for direct steering into lead 2.
Figure 44: Imaging with $T_{12}$ and $T_{32}$. "Rebound" portions of $T_{12}$ and $T_{32}$ plotted versus angle of emission. Top: Corresponding simple classical electron trajectories
the beam is imaged (i.e. scattered into the opposite side probe). Changing the sign of the magnetic field, the other half of the angular profile appears in the other side probe. In this case, a different reflecting boundary is involved so that $\pm B$ differences are expected. Figure 44 is a plot of these “rebound” features as a function of angle of emission, $\theta$, from point contact 2. A trajectory analysis discussed in Appendix A yields the function $\theta(X^+, r_c)$, which is used to convert field into angle. It requires an additional parameter $X^+ (X^-)$ which is the distance of the left (right) reflection point from the center of the collector point contact. Values of $X^+$ and $X^-$ used to obtain the curves of Figure 44 are noted in Table 2. They are determined by fitting the main “rebound” peaks to a zero angle of emission. The curves of Figure 44 show the same side-lobe features observed in Figure 40. A consistent picture emerges, despite the fact that quantitative comparisons are difficult because of the complexity of the collection within the “rebound” geometry. Furthermore, the hard-wall approximation only qualitatively represents the true, more rounded, potential contours. The presence of the same fine structure in $T_R$ and $T_L$ as observed in $\hat{T}_R$ and $\hat{T}_L$ follows from reciprocity and is due to the angular dependence of collection in the former case rather than of injection in the latter case.

6.4.3 Details of calculating $dT/d\theta$

Figure 40 presents curves showing the shape of the momentum distribution ($dT/d\theta$) injected by the quantum point contact. Although propagation within the junction is essentially classical, transport at the quantum point contacts is not. They are constricted to the point where, at most, only a few modes propagate. In this quantum mechanical regime, side lobes appear in the emerging beam. The first possibility to consider is that these side-lobe features can be attributed to secondary diffraction maxima.

For a single mode of propagation, one can achieve a collimated central beam with the adiabatic potential shown in the inset of Figure 45. A wire in which only a single mode propagates at the Fermi level widens and then empties into a wide 2DEG region.
Figure 45: $dT/d\theta$ for an adiabatic constriction with a single propagating mode. The center of the constriction is defined by a potential of height $0.38 E_F$ and width $0.95 \lambda_F$. The width and potential are graded to $3.2 \lambda_F$ and zero, respectively, over a distance of greater than $9 \lambda_F$. Inset: Adiabatic point contact potential for the calculation.
The condition for spatial adiabaticity is given by

\[ dW/dx < \frac{1}{N(x)} \]  

(141)

where \( N(x) \approx k_F W(x)/\pi \) and \( W(x) \) is the (transverse) width at (longitudinal) position \( x \)[115]. The injected distribution in this case is shown in Figure 45. Calculations are performed using a recursive Green's function technique[73, 79] as discussed in Appendix B. Secondary diffraction maxima are evident at \( \theta = 40^\circ \) and \( \theta = 65^\circ \) in the calculation, but they are orders of magnitude below the central maximum at \( \theta = 0 \). To compare these results with experiment, \( T_{42} \) is plotted in polar form, employing a logarithmic radial coordinate, in Figure 46. Comparing with Figure 45, it is immediately evident that diffraction cannot explain the side-lobe features observed in the experimentally-imaged distribution, which are less than an order of magnitude smaller than the central maximum.

On the other hand, features of this magnitude can result from injection of a multimode distribution into a semi-infinite (2D) region. Momenta that would be generated by such a wire emptying into a wide 2DEG region are shown in Figures 47b and 47c as allowed points on the semicircle \( |k| = k_F \). Thus, a wire injects modal features at specific angles into a 2D region.\(^4\)

The data, however, show side lobes even for bias \( a \), i.e. \textit{even when only the lowest transverse mode propagates through the constriction}. This requires that the point contact potential be non-adiabatic, as in the case of the model potential in Figure 47d. Here the model potential includes a narrow saddle point (region I) which defines the conductance. The potential violates the condition of Equation 141 and is nonadiabatically graded (through region II) into a wider transition region (region III) which then empties into the wide 2DEG. In both regions I and III, momenta in the transverse direction are quantized in units of \( \pi/W \). In the case of region III, each mode produces a feature in the emitted distribution centered at

\[ \theta_{n}^{(\pm)} = \pm \sin^{-1} \left( \frac{n\pi}{k_F W_{\text{III}}} \right) \]  

(142)

\(^4\)One might expect to see similar features in the electron beam injected by a field-emission tip terminated by a single atom[116, 117].
Figure 46: $T_{42}$ as a function of angle of emission at biases $a$, $b$, and $c$. The side lobes are less than an order of magnitude down from the central peak.
Figure 47: Critical path model of a quantum point contact. (a) The depletion region defining the effective boundaries of the constriction (shaded) do not follow the smooth contour of the gates themselves. (b) Momenta on the $|k| = k_F$ shell that would be emitted from region III. (c) Momenta on the $|k| = k_F$ shell that would be emitted from region I. (d) Non-adiabatic model potential. The narrowest region (I) of width $W_I$ and potential $V_I$ empties into the transition region (III) or width $W_{III}$ and length $L_{III}$. These are connected by a non-adiabatic region (II), of length $L_{II}$, with width and potential both linearly graded to match at regions I and III.
The angular spread produced by each transverse mode is determined by diffraction. The details of the intermode scattering determine the amount of flux from each occupied mode of region I which is ultimately transferred into the occupied modes of region III. Electrons that pass from region I to region III in a given mode $n$ become collimated. Intermode scattering, in general, will not affect all electron in a given mode. For those electrons which remain in the same mode as they pass into region III, the increased width of III lowers their transverse momentum, $k_{\parallel}^{(n)}$, thereby increasing their forward momentum,

$$k_{\parallel}^{(n)} = \sqrt{k_F^2 - k_{\perp}^{(n)}^2}$$  \hspace{1cm} (143)

Nixon and Davies\[118\] have found large potential fluctuations for quantum constrictions defined on GaAs/AlGaAs modulation-doped 2DEG heterostructures. These fluctuations result from the random positions of the ionized donors within the doped $Al_{0.3}Ga_{0.7}As$ layer\[89\]. Potential variations which are large fractions of the Fermi energy can occur on length scales comparable with the Fermi wavelength. Near the edges of the constriction where the potential rises to meet the Fermi energy, these fluctuations are most pronounced. Thus, the depletion region defining the effective boundaries of the constriction (shaded) will not follow the smooth contours of the gates themselves. In a short ballistic constriction, this will in general result in a conductance controlled by a single, low-density, narrow region – a critical path. The model potential of Figure 47d simulates this situation.

This potential is now used in additional recursive Green’s function calculations. The narrow region (I) of width $W_I$, potential $V_I$, and length $L_I = 1.6\lambda_F$ empties into the transition region (III) of fixed width $W_{III} = 3.2\lambda_F$, length $L_{III} = 1.6\lambda_F$, and potential $V_{III} = 0$. The nonadiabatic region (II) has fixed length, $L_{II} = 1.6\lambda_F$, with width and potential both linearly graded to match at regions I and II. Figure 48a is a plot of the calculated flux distribution $dT/d\theta$ for one, two, and three modes of propagation through the saddle-point (labelled $a$, $b$, and $c$, respectively). For curves $a$, $b$, and $c$, $W_I = 0.95, 1.3$, and $1.6\lambda_F$, while $V_I = 0.38, 0.04$, and $0.00E_F$, respectively. These calculations are compared with the experimental results for $dT/d\theta$ in Figure 40. One observes strong qualitative agreement. The calculated results of Figure 48b show
Figure 48: (a) $dT/d\theta$ for the non-adiabatic point contact potential of Figure 45 with $L_{II} = 0.8\lambda_F$, $W_{III} = 3.2\lambda_F$, and $L_{III} = 1.6\lambda_F$. Curves a, b, and c correspond to 1,2, and 3 injected modes, respectively. For curve a, b, and c, $W_I = 0.95$, 1.3, and $1.6\lambda_F$, respectively; while $V_I = 0.38$, 0.04, and $0.00E_F$, respectively. (cf. the experimental results of Figure 40) (b) Modal decomposition of curve c into even (dotted) and odd(dashed) components. Calculated angular peak positions for transverse modes 1 through 5 of region III are labelled.
the modal decomposition of curve c into even and odd parity components. These are separately conserved because of the mirror-plane symmetry of the model potential of Figure 47d. Calculated angular peak positions for transverse modes 1 through 5 of region III are labelled. For curve c of Figure 48a, the predominant "shoulders" of the sum curve arise from intermode scattering into the fourth and fifth modes. For curve a, analogous "shoulders" arise solely due to scattering from mode 1 into mode 3.

The additional calculations presented in Figure 49 show the importance of the transition region (region III). Values of potential and Fermi energy are utilized which result in a single mode of propagation through the saddle-point \((V_I = 0.38E_F, W_I = 0.95\lambda_F)\). The length of the transition region is progressively decreased from \(2.4\lambda_F\) to 0, while the length of the non-adiabatic region is held at fixed length, \(L_{III} = 1.6\lambda_F\). As \(W_{III}\) is reduced, diffraction broadens the angular spread of the modal features of \(dT/d\theta\) until they coalesce and are lost in a single broad peak[119].

Finally, the importance of the non-adiabatic region in mode conversion is shown by considering the one to three mode conversion observed as the shoulder in the emitted flux distribution for curve a of Figure 48a. Increasing the length of region II decreases the amount of nonadiabaticity within the model potential. Figure 50 is a plot \(dT/d\theta\) for the non-adiabatic region lengthened from \(L_{II} = 0.8\lambda_F\) to \(L_{II} = 2.4\lambda_F\). Comparing with Figure 49b, one can observe the corresponding decrease in intermode scattering from mode one to mode three.

This model of a short quantum point contains a non-adiabatic[120] potential. This is completely consistent with the conductance quantization observed in Figure 38 as long as intermode scattering can occur within these point contacts without significant backscatter. Calculations using realistic point contact potentials comprising a single region of constricted width and density appear to confirm these ideas[121]. Conductance quantization is lost, however, when more than one constricted region is present within the point contact. Backscatter can now be enhanced by resonance effects, a possibility that becomes greater with increasing point contact length[118]. Furthermore, boundary scattering effects begin to play a role for length \(> 0.5\mu m[85]\).

To demonstrate that this single-critical-path, nonadiabatic model potential preserves conductance quantization, consider the point contact potential used in case b
Figure 49: Importance of the transition region. \(dT/d\theta\) for the case of a single mode propagating through the saddle point \((W_I = 0.95\lambda_F, V_I = 0.38E_F)\). The non-adiabatic region is of fixed length \(L_{III} = 1.6\lambda_F\). Left inset of each plot is a corresponding polar representation. Right inset of each plot is a representation of the model potential. \(W_{III}\) is successively reduced. (a) \(W_{III} = 2.4\lambda_F\) (b) \(W_{III} = 1.6\lambda_F\) (c) \(W_{III} = 0.8\lambda_F\) (d) \(W_{III} = 0\).
Figure 50: Importance of the non-adiabatic region. $dT/d\theta$ for the case of a single mode propagating through the saddle-point ($W_I = 0.95\lambda_F$, $V_I = 0.3SE_F$). (cf. Figure 49b) In this case, $L_{II} = 2.4\lambda_F$, $L_{III} = 1.6\lambda_F$. Increasing the length of region II decreases intermode scattering. *Left inset:* Corresponding polar representation. *Right inset:* Representation of the model potential.
of Figure 48a. Figure 51 is a plot of the conductance of the constriction as a function of the saddle-point potential $V_o$. As $V_o$ is increased from 0 to $E_F$, the conductance shows quantized plateaus.

![Figure 51: Non-adiabaticity and conductance quantization. Model potential calculation with $W_I = 1.3\lambda_F$, $L_I = 1.6\lambda_F$, $V_I = V_o$, $L_{II} = 1.6\lambda_F$, $W_{III} = 3.2\lambda_F$, $L_{III} = 1.6\lambda_F$. Conductance plotted as a function of $V_o$ for $V_o$ varied from 0 to $E_F$. At $V_o = 0.04E_F$, this is the potential used for curve b in Figure 48a.]

6.4.4 Reproducibility of quantum point contact potentials

In the model proposed above, features in the injected flux distribution are attributed to details in the potentials of the quantum point contact. In particular, the side lobes of the distribution are expected to be very dependent on intermode scattering, which is, in turn, quite dependent on nonadiabaticity of potentials.

This method of extracting the transmission coefficients is accomplished by means of a least-squares fit to an overdetermined linear system as described in Section 6.2.1.
This permits one to study fluctuations in the quantum point contact potential that occur during the four or more hours required to complete the measurement.

The variance in $T_{42}$ is plotted in Figure 52 along with the least-squares fit, $\tilde{T}_{42}$, for bias $c$. The error is always $\leq 5\%$. The interesting point to note is that the variance is peaked at magnetic fields corresponding to the shoulders in $T_{42}$, that is, to those parts of the distribution corresponding to higher-order mode injection. It is exactly these features in the curve that we expect to be most sensitive to changes in the point contact potential. Similar peaks in the variance are found for the other transmission coefficients at magnetic fields corresponding to transmission of side-lobe features. These are most likely discrete changes in trapped charge in the $AlGaAs$, that occur during the course of the four or more hours required to take the seven magnetic field sweeps. During this time, gate bias and temperature are held fixed.

In addition, the point contact potentials show poor reproducibility with thermal cycling. Figure 53 shows $T_{42}$ on two successive cooldowns for a single device. Bias points $a$, $b$, and $c$ are chosen to yield the same quantized ballistic resistance in each case. The potential within the point contact has changed on thermal cycling, although modal features are present in both sets of curves. This variation is also indicative of variations observed from device to device.

### 6.4.5 Weakly-coupled probes

In the case of a ballistic four-probe junction, strong coupling is always obtained when, at the junction, little or no geometrical distinction exists between voltage and current probes. This is exemplified by the symmetric cross junction of Figure 31. There has been considerable theoretical interest in weakly-coupled probes as a means of “noninvasively” measuring the voltage drops within small conductors. To realize this opposite extreme requires voltage probes almost totally decoupled from the main current path. Engquist and Anderson[122] define a weakly-coupled probe as accepting only infinitesimal transmission, $t$, compared to transmission along the main conductor itself, $T$; that is, $t/T \ll 1$. In the pinched cross junction, weak coupling is actually achieved by using the gated constrictions to separate two of the probes from the main channel.
Figure 52: Variance in $T_{42}$ for the *pinched* junction. (a) $T_{42}$ for bias $c$ as determined by the least squares fit. (b) Corresponding variance $\sigma_{T_{42}}^2$. The variance is peaked at magnetic fields corresponding to those parts of the curve associated with higher-order mode injection.
Figure 53: Reproducibility of point contact potential with thermal cycling. (a) and (b): $T_{42}$ measured on two successive cooldowns. Bias points $a$, $b$, and $c$ are chosen to yield the same quantized ballistic resistance in each case.
Figure 54 shows the reconstituted Hall resistance for the pinched junction, $R_H(B) = R_{13,24}(B)$, as well as the measured value for bias point $c$. As in the case of the bend

![Graph showing Hall resistance vs. magnetic field](Image)

Figure 54: Hall resistance, $R_H = R_{13,24}(B)$, for the pinched cross junction at biases $a$, $b$, and $c$ reconstituted from the measured $T_{ij}$ (solid curves). Also shown is $R_H$ measured conventionally at bias $c$ (dashed curves). Curves for $a$ and $c$ are offset by 0.1 and -0.1, respectively. *Inset:* Choice of current and voltage probes for this measurement.

resistance measurement, weak coupling precludes this four-terminal resistance measurement for biases $a$ and $b$. The quenched Hall effect reflects the rebound features observed in turning transmission coefficients. Most surprising is that this effect grows as the probes become decoupled. As noted previously, with increasing bias, the anomalous rebound peaks acquire similar magnitude to the total flux ultimately collected by the “correct” probe at higher $B$. The quench of $R_H$ is thereby enhanced by progressive probe decoupling.
Büttiker has applied his four-terminal resistance formalism to consider the special case in which transmission into two of the probes, \( t \), is weak\[123\] The transmission coefficients for the structure can then be expanded with respect to the small parameter \( t \). To lowest order, \( T_{31} = T_{31}^{(0)} \), \( T_{21} = t T_{21}^{(1)} \), \( T_{43} = t T_{43}^{(1)} \). Similar expansions follow for each transmission coefficient. Büttiker then obtains the formula:

\[
R_{13,24} = \frac{\hbar}{2e^2 T} \frac{T_{21}^{(1)} T_{43}^{(1)} - T_{23}^{(1)} T_{41}^{(1)}}{(T_{21}^{(1)} + T_{23}^{(1)})(T_{41}^{(1)} + T_{43}^{(1)})}
\]  \hspace{1cm} (144)

where \( T = T_{31}^{(0)} = T_{13}^{(0)} \). The factor

\[
f = \frac{T_{21}^{(1)} T_{43}^{(1)} - T_{23}^{(1)} T_{41}^{(1)}}{(T_{21}^{(1)} + T_{23}^{(1)})(T_{41}^{(1)} + T_{43}^{(1)})}
\]  \hspace{1cm} (145)

is what is identified by Peeters\[68\] and Akera and Ando\[124\] as the geometrical form factor describing how the Hall resistance vanishes with decreasing magnetic field. This clearly depends on the details of the coupling.

Using the experimental results of Figures 42 and 43, the form factor for bias \( a \) can be extracted from Equation 145. The results are shown in Figure 55a. This result differs considerably from previous calculations. Monotonically decreasing form factors have been predicted in previous theoretical calculations. These involve simple models in which the weak coupling is achieved by a tunnel barrier in which the transmission is proportional to \(|\partial \Psi_o / \partial x|^2\), where \( \Psi_o \) is the unperturbed wire wavefunction and the derivative is evaluated at the wire edge where the voltage probe is attached. The form factor calculated in Figure 55 instead shows structure due to the details of the momentum distribution injected by the weakly-coupled probes and the exact nature of the scattering from the potentials induced in the neighborhood of the probes.

Figure 55b, also shows the transmission coefficient, \( T_F = T_{31} \), as a function of field. It is relatively constant except for small dips centered at \(|B| \sim 0.3T\). This is probably due to the enhanced backscatter, within this field range, of certain trajectories from the slight distortions of the main channel in the vicinity of the side probes. In this structure, collimation appears to play a minor role in the generation of the quench, contrary to results of simulations reported in Baranger and Stone\[125\].
Figure 55: Form factor (see Equation 145) and $T_{31}$ as a function of magnetic field for bias a.

6.5 Summary

In ballistic semiconductor structures, a single dominant scattering region can be isolated and probed with quasi-one-dimensional leads. A new experimental technique is presented for measuring the transmission coefficients in specially-designed ballistic multiprobe microstructures. It is founded upon theoretical work which treats transport in these structures as a scattering problem.

The scattering properties of a open cross junction are measured. There are strong features in the coefficients for turning which can be attributed to simple "rebound" trajectories. These transmission coefficients can be used to successfully calculate any four-terminal resistance.

This chapter has also included a detailed study of the pinched cross junction, in which two of the probes are separated from the main channel by quantum point contacts. This series arrangement of point contacts allows a sensitive momentum
spectroscopy of the emitted distribution, even in the case of nearly vanishing electron flux. Although propagation within the junction is essentially classical in these experiments, transport at the quantum point contact is not. In the limit at which only a few modes propagate, one observes side lobes in the emerging beam. This is most evident in the transmission coefficients for propagation through both point contacts. "Rebound" features persist in the turning coefficients for this geometry and show structure due to specular scattering of the side lobes. The side lobes are modal features in the outgoing distribution, which are present even in the case of a single propagating mode. This implies that quantum point contact potentials are, in general, non-adiabatic, even in the presence of conductance quantization. For short contacts, which demonstrate conductance quantization, transport is dominated by a single critical path of constricted width and density defined by potential fluctuations produced by the random position of ionized donors. This pinched geometry also provides the first fully-characterized realization of weakly-coupled probes. Signatures of junction scattering persist in this case and are, in fact, strengthened in the limit of weak coupling.
Chapter 7

Conclusions

7.1 Summary of contributions

This dissertation has made several contributions to the understanding of electron transport in mesoscopic systems:

- I have presented a simple linear response derivation of the generalized multichannel, multiprobe Landauer formula, derived by Büttiker for spinless, noninteracting electrons. It is the first such calculation to make use of the analytic properties of the $S$-matrix elements[59]. This derivation has been used by others as a basis for calculations of the current response of phase-coherent conductors to oscillating external electric fields[126].

- I have developed an extension of the recursive Green's function technique to high magnetic fields, which allows numerical calculation of the transmission coefficients for an arbitrary two-reservoir phase-coherent conductor. Using this technique, I study the suppression of backscatter as the mechanism for the transition from diffusive to high-field transport in quantum wires[73]. The quantum Hall regime is entered when there is negligible overlap between electron edge states localized at opposite sides of the wire. Levitation of the quantized plateaus with increasing disorder is observed.
CHAPTER 7. CONCLUSIONS

- I have studied exchange-enhanced spin splitting of Landau levels within quantum wires, a problem that requires consideration beyond the context of the independent electron approximation[94]. I find that this spin-splitting, as evidenced in two-terminal conductance measurements, is increased by the confinement. This is due to an effect dubbed "antiscreening", since electron-electron interactions are strengthened by interaction with polarizable edge-state charge.

- I have developed a general experimental approach yielding the entire transmission matrix of a multiprobe mesoscopic conductor[105, 106]. Using a sample design which is an almost literal realization of the Landauer-Büttiker model, results are presented for several new investigations enabled by this technique. The transmission coefficients for a simple cross junction are measured, and it is easily verified that these coefficients reproduce the measured Hall and bend resistance low-field anomalies. In this sense, it represents a definitive study of classical ballistic transport in junctions. The other investigations center on a junction in which two of the probes are separated from the main channel by quantum point contacts. This series arrangement of point contacts allows a sensitive momentum spectroscopy of the emitted distribution. Even in the case of a single propagating mode, surprising modal features in the outgoing distribution are observed. This indicates new details about the potentials within quantum point contacts. In particular, it indicates that conductance quantization can be present even in the presence of non-adiabatic potentials and significant mode conversion. This pinch geometry also allows the first fully-characterized realization of weakly-coupled probes.

7.2 Future electronic devices

I began this dissertation with the prediction that as scaling of electron devices continues, transport will eventually impinge on the mesoscopic regime. I would like to conclude with a discussion of how mesoscopic transport phenomena might be applied to produce novel device operation, with an emphasis on logic and memory applications. I will focus on four transport phenomena, two of which show promise. One of
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the key features in many of these ideas is multi-functionality, in which an operation requiring several interconnected transistors can be accomplished with a single device. In the quest for high speed and low power (low switching energy), one must not lose sight of the often overlooked characteristics of a good computer device[127]: fan-out capability, high gain which allows the device to operate in an environment without large precision or reproducibility, and good isolation of input and output.

7.2.1 Quantum confinement effects

The quantized ballistic resistance studied in Chapter 6 is one notable example of a quantum-size effect, an effect associated with the discrete energy levels produced by confinement. Device designs based on tunneling through such a discrete energy level spectrum are numerous.\(^1\) The resonant tunneling diode, for example, shows a negative differential resistance due to tunneling through a quantum-confined state at temperatures as high as 300\(K\). There is, however, no easy way to change energy level spectra with a third terminal and, therefore, produce multi-terminal device operation. Quantum-size effects are also, in general, not very robust or reproducible because of extreme sensitivity to changes in the confinement potential.

7.2.2 Phase coherence effects

In devices cooled to sufficiently low temperatures (\(T < 1.5K\)) that acoustic-phonon scattering and electron-electron scattering are frozen out, it is possible to achieve phase-coherent transport. A solid-state electron optics has been proposed using potentials to change the “index of refraction” to create lenses[129, 130]. The requirement here is that temperatures are low enough that electrons are phase coherent on a length scale larger than the size of the lens. Other proposals include devices in which “transistor” action is achieved with interference of electron waves within “waveguide” geometries.[131] This requires phase coherence across the entire extent of the device. Like quantum confinement effects, phase-coherent effects are not very

\(^{1}\)For a review, see Weisbuch and Vinter[128].
robust or reproducible and have very limited potential applicability because of the low temperatures required for operation.

7.2.3 Ballistic/deterministic electron dynamics

One promising possibility is to make use of ballistic electron dynamics in nanometer-scale geometries. One can think of multifunctional device designs based on steering a ballistic electron beam. This steering can be accomplished by using a gate to modify the potential within the device, achieving isolation of input and output comparable with traditional field-effect devices. Because one can tolerate phase-breaking (inelastic) collisions as long as they have little randomizing effect on momentum, reasonably deterministic operation with high gain might be possible at temperatures up to 77 K. Application of ballistic effects to devices requires a more detailed understanding of nonlinear transport.

7.2.4 Single-electron tunneling devices

One effect not considered in this dissertation which may have considerable promise and invites further consideration is the Coulomb blockade of single-electron tunneling [132]. The basic device is the electromagnetic dual of the DC SQUID, and it has been demonstrated using both metals [133] and semiconductors [134]. Switching energies are very low (~ $10^{-22}J$ for capacitance of $50aF$) since the device can be turned on and off with the charge of a single electron. One should contrast this with the 10,000 or more electrons required to define a “1” in a one-transistor DRAM cell. In addition, the technology can be scaled to macromolecular dimensions without fundamental limitations. The characteristic energy gap is given by the single electron charging energy $e^2/2C$. For a $30 \times 30nm^2$ junction (one of the smallest successfully fabricated), $e^2/2C \sim 30K$. Low temperature operation is required, which has, in fact, limited experiment work to dilution refrigerator temperatures. Like Josephson junction technology, single electron tunneling devices generally have poor gain; therefore,

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2In evaluating the potential of single-electron devices, the dual relationship to Josephson-junction logic proves useful. Review of Josephson-junction technology can be found in Gheewala[135] and Van Duze[136].
logic schemes that circumvent this problem are required. In addition, characteristic impedances for these devices are quite high \((> \frac{4}{5})\) and not matched to typical transmission line impedances; therefore, long interconnections must be avoided.
Appendix A

"Imaging" by Rebound Trajectories

This Appendix describes the details of calculating semiclassical propagation of electrons in a magnetic field from an injector to a reflection point a distance $X^+$ away from center.

For $X^+ < X^+_{\text{crit}}$, refer to Figure 56a. Introducing the angle $\alpha$, the relevant relations are:

$$\tan(\alpha + \theta) = \frac{X^+}{L - 2r_c \sin \theta}$$  \hspace{1cm} (146)

$$X^+^2 + (L - 2r_c \sin \theta)^2 = 4r_c^2 \sin^2 \alpha$$  \hspace{1cm} (147)

where $r_c = (h/eB)\sqrt{n_s/2\pi}$ is the cyclotron radius. $L$ is the distance between the collector and injector. These relations can be solved to find $\theta(X^+, r_c)$, the angle of emission reflected by the reflection point at $X^+$ at a given magnetic field. $X^+_{\text{crit}}$ is defined such that $\theta(X^+_{\text{crit}}, r_c) = 0$.

For $X^+ > X^+_{\text{crit}}$, refer to Figure 56b. In this case, the relevant relations are:

$$\tan(\alpha + \theta) = \frac{X^+}{L}$$  \hspace{1cm} (148)

$$X^+^2 + L^2 = 4r_c^2 \sin^2 \alpha$$  \hspace{1cm} (149)

which can be used to find $\theta(X^+, r_c)$. 

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Figure 56: Classical trajectories for momentum spectroscopy by "rebound". (a) Trajectory for \( X^+ < X^+_{\text{crit}} \) (b) Trajectory for \( X^+ > X^+_{\text{crit}} \)
Appendix B

Calculating $dT/d\theta$

This Appendix describes the details of using the recursive Green's function calculations of Chapter 3 to compute momentum distributions emitted from quantum point contacts. The model potential is discretized into lattice points as shown in Figure 57. Semi-infinite ideal leads 100 lattice constants wide are attached at the left and right to approximate wide 2DEG regions. The transmission coefficients $|t_{ij,mn}|^2$ are calculated between planes 50 lattice constants away from the constriction. The momentum distribution in this "far-field" limit (many Fermi wavelengths away from the point contact) can be computed. First, note that in the very wide wires to the right and left of the constriction, the transverse eigenstates are given by:

$$\chi_{jn} = \left(\frac{2}{W}\right)^{1/2} \sin(k_y y) = \left(\frac{2}{W}\right)^{1/2} \frac{e^{ik_y y} - e^{-ik_y y}}{2i}$$

(150)

where $k_y = n\pi/W$, which for $W$ large is a nearly continuous momentum distribution. Deriving an expression for $dT/d\theta$,

$$\frac{dT}{d\theta} = \frac{1}{2\pi} \sum_m |t_{ij,mn}|^2 \left| \frac{dn}{d\theta} \right|$$

(151)

where we treat $n$ as a continuous variable with

$$n = \frac{k_F W}{\pi} \sin \theta$$

(152)

Therefore,

$$\frac{dT}{d\theta} = \frac{k_F W}{2\pi^2 \cos \theta} \sum_m |t_{ij,mn}|^2$$

(153)

This expression is used in the calculations of Section 6.4.3.
Figure 57: Discretization of the point contact potential into lattice points for recursive Green's function calculations. Semi-infinite ideal leads 100 lattice constants wide are attached at the left and right to approximate wide 2DEG regions.
Bibliography


